Search-and-matching: The Mortensen-Pissarides model

Sergio Feijoo-Moreira*
(based on Matthias Kredler's lectures)

Universidad Carlos III de Madrid

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Abstract

These are notes that I took from the course Macroeconomics II at UC3M, taught by Matthias Kredler during the Spring semester of 2016. Typos and errors are possible, and are my sole responsibility and not that of the instructor.

Contents

*Address: Universidad Carlos III de Madrid. Department of Economics, Calle Madrid 126, 28903 Getafe, Spain. E-mail: sfeijoo@eco.uc3m.es. Web: https://sergiofeijoo.github.io.

1 Environment

These notes are mainly based on (Ljungqvist and Sargent, 2012, pp.1135-1142). The search-and-matching model we review is a model of unemployment within the matching framework, based on Diamond (1982), Mortensen (1982), and Pissarides (1990).

The basic model is constructed as follows: time is discrete and infinite ($t = 0, 1, \ldots$). There is a continuum of infinitely lived risk-neutral workers with a discount factor $\beta = (1+r)^{-1}$ and with measure normalized to 1. The objective of each worker is to maximize the expected discounted value of leisure and labor income. The leisure enjoyed by an unemployed worker is denoted by z, while the current utility of an employed worker is given by the wage rate w. We assume that there is an initial measure $u_0 \in [0,1]$ of unemployed and a large measure $\mathcal{M} > 1$ of potential entrants (firms).

The production technology has constant returns to scale, being labor the only input. Each employed worker produces y units of output. Without loss of generality, we assume that each firm employs at most one worker. The firms have the same discount factor as the workers. A firm entering the economy incurs a vacancy cost c in each period when looking for a worker, and when matched, the firm's per-period earnings are $y - w$. All matches are exogenously destroyed with constant exogenous probability $\delta \in (0,1)$. Moreover, each worker and each firm have the same probability of being matched.

The measure of successful matches in a period is given by an ad-hoc matching function $M(u, v)$, where u and v are the aggregate measures of unemployed workers and posted vacancies. This functions satisfies the following properties:

- It is increasing in both u and v.
- It is concave.
- It is homogeneous of degree 1.

Finally, when a worker meets a firm, the wage rate w is assumed to be determined by Nash-bargaining. We will let $\phi \in [0, 1]$ denote the worker's bargaining strength, or his weight in the Nash product (to be defined later on).

1.1 Matching and labor market tightness

We define the ratio between vacancies and unemployed workers, $\theta \equiv v/u$, as the tightness of the labor market (seen from the point of view of the firm). As the matching function is homogeneous of degree 1 and each firm has the same probability of being matched, we can write the probability of filling a vacancy as

$$\Pr(\text{Fill vacancy}) \equiv \frac{M(u,v)}{v} = \frac{vM\left(\frac{u}{v},1\right)}{v} = M\left(\frac{u}{v},1\right) = M\left(\frac{1}{\theta},1\right) \equiv q(\theta).$$

Moreover, using again homogeneity of the matching function and symmetry across workers, we can write the probability of an unemployed worker finding a job as

$$\Pr(\text{Find job}) \equiv \frac{M(u,v)}{u} = \frac{v}{u}M\left(\frac{u}{v},1\right) = \theta q(\theta).$$

Example 1.1 (Cobb-Douglas matching function). *Consider the matching function*

$$M(u,v) = Au^{\alpha}v^{1-\alpha},$$

where $A > 0$ *and* $\alpha \in (0,1)$. *In this case we have*

$$\Pr(\textit{Fill vacancy}) \equiv \frac{Au^{\alpha}v^{1-\alpha}}{v} = Au^{\alpha}v^{-\alpha} = A\left(\frac{v}{u}\right)^{-\alpha} = A\theta^{-\alpha} \equiv q(\theta),$$

and

$$\Pr(\textit{Find job}) \equiv \frac{Au^{\alpha}v^{1-\alpha}}{u} = A\left(\frac{v}{u}\right)^{1-\alpha} = A\theta^{1-\alpha} = \theta q(\theta).$$

2 Value functions (in stationary equilibrium)

Take θ and w such that $\theta_t = \theta$, $w_t = w$, $\forall t$, as given. Let $U \in \mathbb{R}$ be the value of an unemployed worker and let $W \in \mathbb{R}$ be the value of an employed worker. Then, U is given by

$$U = z + \beta[(1-\theta q(\theta))U + \theta q(\theta)W], \tag{U}$$

and W is given by

$$W = w + \beta[\delta U + (1-\delta)W]. \tag{W}$$

Furthermore, let $V \in \mathbb{R}$ be the value of an unmatched firm (the value of a vacancy), and let $J \in \mathbb{R}$ be the value of employing a worker (value of a filling a vacancy). Then V is given by

$$V = \max\{\underbrace{-c + \beta[q(\theta)J + (1 - q(\theta))V]}_{\text{Posting a vacancy}}, \underbrace{0 + \beta V}_{\substack{\text{Not}\\\text{posting}}}\}, \tag{$\tilde{V}$}$$

and J is given by

$$J = y - w + \beta[\delta V + (1 - \delta)J]. \tag{J}$$

Proposition 2.1 (Free-entry condition). *Consider a firm that at time t is deciding whether to post a vacancy. From ($\tilde{V}$), let us define V_p as the value of posting a vacancy and V_n as the value of not posting a vacancy, which are given by*

$$\begin{aligned} V_n &= 0 + \beta V, \\ V_p &= -c + \beta[q(\theta)J + (1 - q(\theta))V]. \end{aligned}$$

As there is a large number of potential entrants, some will find it optimal not to enter. In equilibrium, a firm must be indifferent between posting a vacancy and not posting it, which implies that $V_n = V_p$. In this case,

$$V = \max\{V_p, V_n\} = \max\{V_n, V_n\} = V_n = \beta V,$$

which implies that $V = 0$. In other words, free entry implies that the expected discounted stream of a new firm's vacancy costs and earnings is equal to zero, i.e.

$$V = 0. \tag{FEC}$$

As a consequence, by indifference between posting and not posting, from ($\tilde{V}$) we then have

$$V = -c + \beta[q(\theta)J + (1 - q(\theta))V], \tag{V}$$

where substituting $V = 0$ gives

$$0 = -c + \beta q(\theta)J,$$

which can be rewritten as

$$c = \beta q(\theta)J. \tag{1}$$

From (J) note that as $V = 0$ we can write

$$J(1 - \beta(1 - \delta)) = y - w,$$

where rewriting and substituting $\beta = (1+r)^{-1}$ gives

$$J = \frac{y-w}{1-\frac{1-\delta}{1+r}} = \frac{y-w}{\frac{1+r-1+\delta}{1+r}} = \frac{1+r}{r+\delta}(y-w), \tag{2}$$

which can be interpreted as the current value of the firm's surplus.

Now, combining (1) and (2) gives

$$c = \beta q(\theta)\frac{1+r}{r+\delta}(y-w) = q(\theta)\frac{1}{r+\delta}(y-w),$$

which can be rewritten as

$$w = y - \frac{r+\delta}{q(\theta)}c, \tag{JC}$$

which we call the job creation curve. The wage rate determined by this equation ensures that firms that post vacancies break even in an expected present-value sense.

To get a feeling on how the job creation curve looks like, Figure 1 represents it assuming a Cobb-Douglas matching function and parameters $y = 3$, $r = 0.05$, $\delta = 0.04$, $A = 0.15$, $c = 0.75$ and $\alpha = 0.7$.

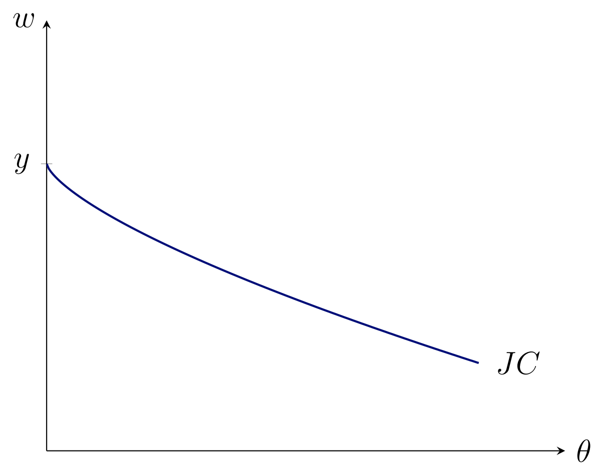

Figure 1: Job creation curve

3 Nash-bargaining

We will take as given (and fixed) everything that occurs after the worker loses or quits his current job - this is summarized in a fixed U. Define the match-surplus for the worker (over his outside option for a given w) as[1]

$$S_w(w) = \tilde{W}(w) - U, \tag{3}$$

[1]Note that the wage in the current match depends on the current negotiation, but not in the negotiation over future matches of the worker.

where

$$\tilde{W}(w) = w + \beta[(1-\delta)\tilde{W}(w) + \delta U].$$

Note that the previous expression can be rewritten as

$$\tilde{W}(w)[1-\beta(1-\delta)] = w + \beta\delta U,$$

where substituting $\beta = (1+r)^{-1}$ we can obtain

$$\tilde{W} = \frac{1+r}{r+\delta}\left(w + \frac{\delta}{1+r}U\right).$$

Therefore we can rewrite (3) as

$$S_w(w) = \frac{1+r}{r+\delta}\left(w + \frac{\delta}{1+r}U\right) - U = \frac{1+r}{r+\delta}\left(w - \frac{r}{1+r}U\right), \tag{S_w}$$

which gives us the flow surplus (differential wage vs. flow value of unemployment). Furthermore, define the match-surplus for the firm (over its outside option for a given w) as

$$S_f(w) = \tilde{J}(w) - V = \tilde{J}(w) = \frac{1+r}{r+\delta}(y-w), \tag{S_f}$$

where the second equality follows from the free entry condition and the third equality follows from (2).

Proposition 3.1 (Generalized Nash-bargaining). *Under the generalized Nash assumptions on the bargaining process (scale invariance, efficiency, and independence of irrelevant alternatives), the wage should satisfy*

$$w^* = \arg\max_w \left\{S_w(w)^\phi S_f(w)^{1-\phi}\right\} \tag{NP}$$

where (NP) is called the Nash product. Intuition: the parameter ϕ gives the workers a share ϕ of the total surplus generated by the match.

Remark. *For $\phi = 0.5$ we have the original 'Nash-bargaining' (which has a symmetry assumption).*

The problem defined by Proposition 3.1 is a bargaining problem with transferable utility, meaning that it is possible to transfer surplus from the worker to the firm (or in the opposite direction) in a one-to-one basis. This can be seen by taking derivatives in the expressions (S_w) and (S_f), as

$$\frac{\partial S_w(w)}{\partial w} = -\frac{\partial S_f(w)}{\partial w} = \frac{1+r}{r+\delta}.$$

The total surplus generated by a match is obtained as the sum of the surplus for the firm and the surplus for the worker, i.e.

$$S_{tot} = S_w(w) + S_f(w) = \frac{1+r}{r+\delta}\left(y - \frac{r}{1+r}U\right).$$

Note that the total surplus in this setup does not depend on the wage, and the simplicity of the previous expression is due to the linear utility assumption.

Proposition 3.2 **(Nash-bargaining solution).** *The Nash-bargaining solution w^* for a problem with transferable utility satisfies*

$$\tilde{S}_w(w^*) = \phi S_{tot},$$
$$\tilde{S}_f(w^*) = (1-\phi)S_{tot}.$$

Proof. Homework ■

Remark. *The parameter ϕ can be understood as a value that captures labor market characteristics. Note that if $\phi = 1$ ($\phi = 0$), the the worker (firm) would be making a take-it-or-leave-it offer to the firm (worker).*

4 Equilibrium

Our aim in this section is pinning down the unique equilibrium of this model. To do this, note that from Proposition 3.2 we can write

$$\frac{\tilde{S}_w(w^*)}{\phi} = S_{tot} = \frac{\tilde{S}_f(w^*)}{1-\phi},$$

which can be rewritten as

$$(1-\phi)\tilde{S}_w(w^*) = \phi\tilde{S}_f(w^*).$$

Substituting (S_f) and (S_w) in the previous expression we obtain

$$(1-\phi)\frac{1+r}{r+\delta}\left(w^* - \frac{r}{1+r}U\right) = \phi\frac{1+r}{r+\delta}(y - w^*),$$

from where we can obtain

$$w^* = \phi y + \frac{r}{1+r}(1-\phi)U = \frac{r}{1+r}U + \phi\left(y - \frac{r}{1+r}U\right), \tag{w^*}$$

which is the wage w^* that is consistent with Nash Bargaining for a given U. Note that this expression is a partial equilibrium object, as it depends on the outside option of the worker. To find an expression that only depends on the labor market tightness, we need to derive an expression for $r(1+r)^{-1}U$. To do so, we will first obtain an expression for $S_w = W - U$ from (U), where we have

$$\begin{aligned} U &= z + \beta[(1-\theta q(\theta))U + \theta q(\theta)W] \\ &= z + \beta\theta q(\theta)(W-U) + \beta U, \end{aligned}$$

which can be rewritten as

$$S_w = W - U = \frac{(1-\beta)U - z}{\beta\theta q(\theta)}.$$

Note that the social welfare for the worker when accepting a match is given by the difference between the value of being employed and her outside option (the value of being unemployed). Substituting $\beta = (1+r)^{-1}$ we obtain

$$S_w = \frac{rU - (1+r)z}{\theta q(\theta)} = \frac{r(U-z) - z}{\theta q(\theta)}. \tag{4}$$

Besides, from Proposition 3.2 we have

$$S_w = \phi S_{tot} = \frac{\phi}{1-\phi} S_f = \frac{\phi}{1-\phi}(J - V) = \frac{\phi}{1-\phi} J,$$

where the last equality follows from (FEC). Furthermore, note that substituting (1) we obtain

$$S_w = \frac{\phi}{1-\phi} \frac{c}{\beta q(\theta)}. \tag{5}$$

Now, equating (4) and (5) we obtain

$$\frac{r(U-z) - z}{\theta q(\theta)} = \frac{\phi}{1-\phi} \frac{c}{\beta q(\theta)},$$

and simplifying the previous expression we can find

$$\frac{rU}{1+r} = z + \frac{\phi}{1-\phi}\theta c, \tag{6}$$

which implies that the flow value of unemployment is not only given by z, the utility from leisure, but also from the option value of finding a job. Substituting (6) in the partial equilibrium wage relationship given by (w^*) we find

$$w^* = z + \frac{\phi}{1-\phi}\theta c + \phi\left(y - z - \frac{\phi}{1-\phi}\theta c\right).$$

Simplifying the previous expression we obtain

$$w^* = z + \phi(y - z) + \phi\theta c, \tag{WC}$$

which is called the wage curve and gives the locus of all (θ, w) combinations that are consistent with a Nash-Bargaining solution. Note that in the previous expression, $z + \phi(y - z)$ gives the static (or myopic) Nash-bargaining wage. The last term tells us that if labor market tightness is high, a worker can find a new job faster when unemployed, which increases her bargaining power when matching with a firm. Figure 2 shows the job creation curve, the wage curve and the equilibrium wage and labor market tightness assuming a Cobb-Douglas matching function and parameters $y = 3$, $r = 0.05$, $\delta = 0.04$, $A = 0.15$, $c = 0.75$, $\alpha = 0.7$, $z = 0.5$ and $\phi = 0.35$.

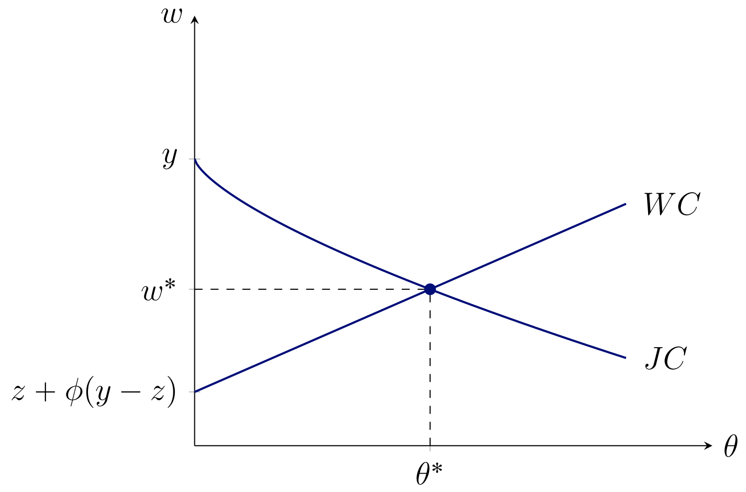

Figure 2: Job creation and Wage curve

Definition 4.1 (Equilibrium). *A stationary (decentralized)*[2] *equilibrium consists of values* $\{U, W, J, V\}$*, quantitites* $\{u^*, v^*, \theta^*\}$ *and a wage* w^* *such that*

1. *The values* U *and* W *satisfy the Bellman equations* (U) *and* (W) *given* θ^* *and* w^**.*
2. *The values* V *and* J *satisfy the Bellman equations* (V) *and* (J) *given* θ^* *and* w^**.*
3. *The value* V^* *is consistent with* (FEC)*.*[3]
4. *The wage* w^* *satisfies* (NP)*, i.e. it is the Nash-bargaining solution given values* U^* *and* V^**.*
5. *Consistency:*
$$\theta^* = \frac{v^*}{u^*}.$$

[2]Note it is not a competitive equilibrium as there is no price-taking behaviour by the agents in this environment.

[3]In other words, firms post vacancies optimally, i.e. $V^* = 0$ holds in the equilibrium.

6. *Unemployment is stationary, i.e. the flows from employment to unemployment are identical to the flows from unemployment to employment:*

$$\underbrace{\delta(1-u^*)}_{\substack{\textit{From employment}\\ \textit{to unemployment}}} = \underbrace{u^*\theta^* q(\theta^*)}_{\substack{\textit{From unemployment}\\ \textit{to employment}}}. \tag{7}$$

Solving for the equilibrium unemployment u^* in (7) we obtain

$$u^* = \frac{\delta}{\delta + \theta q(\theta)}$$

If we consider a Cobb-Douglas matching function, the previous expression becomes

$$u^* = \frac{\delta}{\delta + A\theta^{1-\alpha}}$$

Note that an increase in θ, everything else constant, implies that $\theta q(\theta)$ will also increase, reducing unemployment.

To conclude the characterization of the equilibrium of this model, let us derive an expression that will allow us to pin down the equilibrium value of the labor market tightness. To do so, we equate the Job creation curve (JC) and the Wage curve (WC), obtaining

$$y - \frac{r+\delta}{q(\theta)}c = z + \phi(y-z) + \phi\theta c,$$

which gives[4]

$$F(\theta; y, c, \ldots) \equiv (z-y)(1-\phi) + \phi c\theta + \frac{r+\delta}{q(\theta)}c = 0, \tag{θ}$$

so that we obtain an implicit equation for the equilibrium value of labor market tightness θ^*. Figure 3 plots this function under the same parametrization as before.

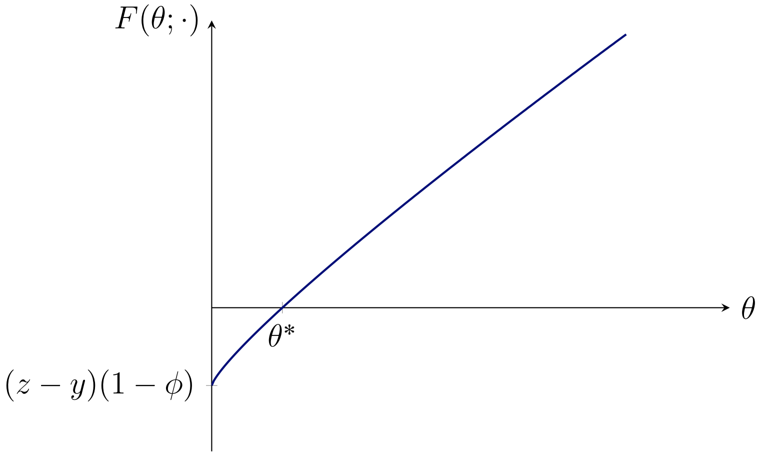

Figure 3: Labor market tightness implicit equation.

[4]Note that there is a unique solution θ^* because $z - y < 0$ and F is strictly increasing in θ.

5 Comparative statics

In this section we perform a comparative statics analysis to understand the effects on the equilibrium wage and labor market tightness when changing deep-model parameters.[5]

5.1 Increase in ϕ

Our first exercise consists on increasing ϕ from 0.35 to 0.55, the results are shown in Figure 4. This change has an effect on the Nash-bargaining surplus that is captured by the workers, and in particular this change effectively moves surplus from the firm to the worker. As we can see, the wage increases. In this case, fewer firms will post vacancies, and the labor market tightness will decrease. As a consequence, the equilibrium unemployment will increase.

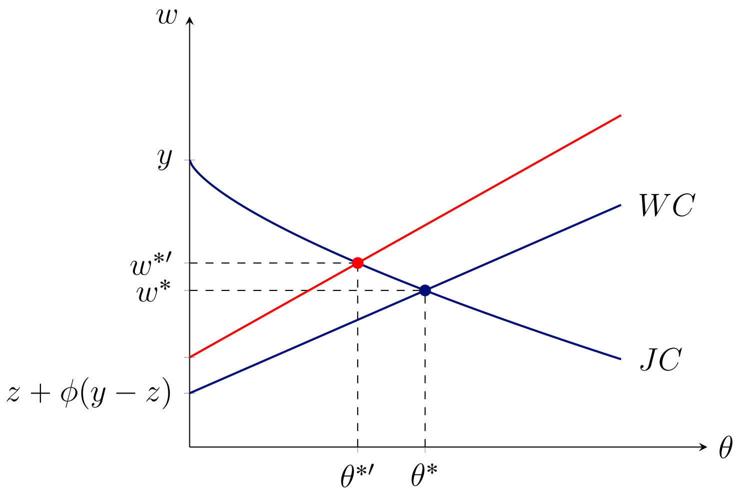

Figure 4: Comparative statics. Effect of increase in ϕ.

5.2 Increase in z

In this case we increase the value of z from 0.5 to 1, the results are shown in Figure 5. This change increases the workers outside option and induces similar effects as when we increase ϕ. As we can see, the wage increases. In this case, fewer firms will post vacancies, and the labor market tightness will decrease. As a consequence, the equilibrium unemployment will increase.

[5]Baseline calibration: $y = 3$, $r = 0.05$, $\delta = 0.04$, $A = 0.15$, $c = 0.75$, $\alpha = 0.7$, $z = 0.5$, $\phi = 0.35$.

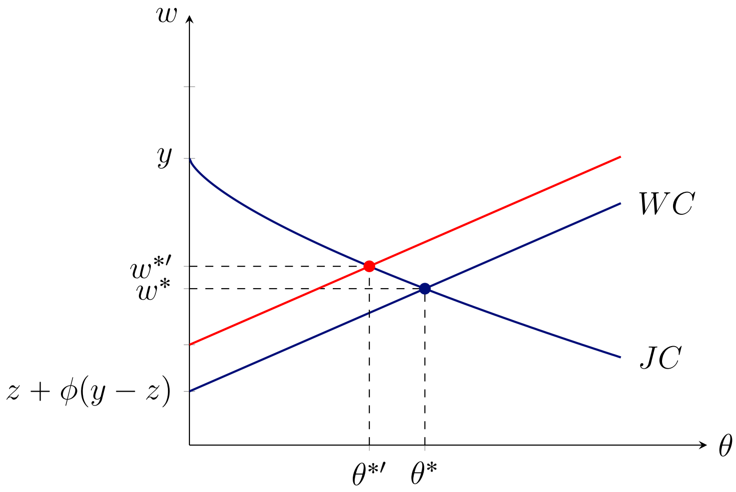

Figure 5: Comparative statics. Effect of increase in z.

5.3 Increase in δ

An increase in δ implies that a higher number of matches are ended in each period, i.e. jobs are destroyed faster. This change induces, ceteris paribus, a decrease in the value of a matched firm. In this case, for a given wage, the labor market tightness must be lower so that firms are able to break even. The equilibrium results are a decrease in the wage and the labor market tightness, and an increase in unemployment. In Figure 6 we show the results obtained when increasing δ from 0.04 to 0.1.

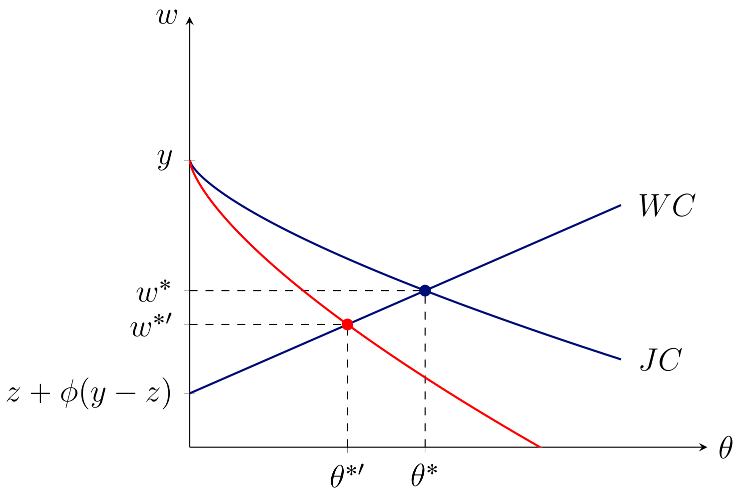

Figure 6: Comparative statics. Effect of increase in δ.

5.4 Increase in y

An increase in y may be understood as business cycles, for example an increase in productivity. In principle, as both curves shift upwards, the equilibrium wage will always be higher, while the effect on the equilibrium labor market tightness is not immediately obvious. We can apply the implicit function theorem on (θ) which allows us to write

$$F_y(\theta; y, c, \dots) \,\mathrm{d}y + F_\theta(\theta; y, c, \dots) \,\mathrm{d}\theta = 0,$$

so that

$$\frac{\mathrm{d}\theta}{\mathrm{d}y} = \frac{-F_y(\theta; y, c, \dots)}{F_\theta(\theta; y, c, \dots)} = \frac{1-\phi}{c\phi - \dfrac{r+\delta}{[q(\theta)]^2} cq'(\theta)}.$$

As $q'(\theta) < 0$, then the equilibrium θ will increase. The results of an increase in value of y from 3 to 3.5 are shown in Figure 7.

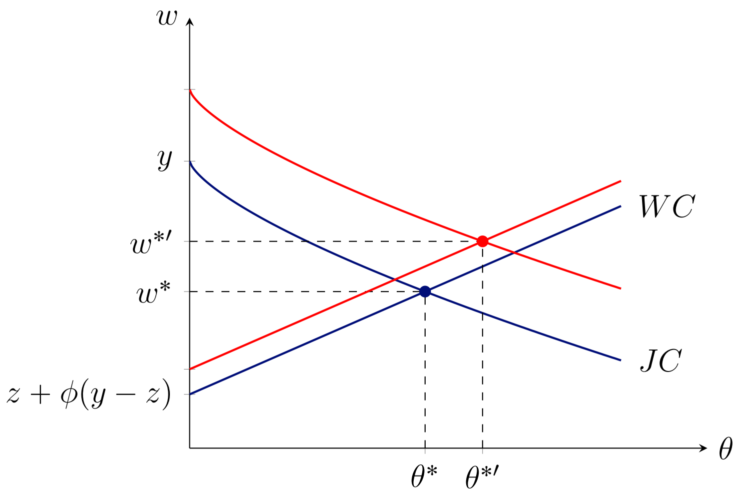

Figure 7: Comparative statics. Effect of increase in y.

5.5 Increase in c

In out final exercise, we increase the value of c from 0.5 to 1. Again, as the slopes of both curves increase in absolute value, the equilibrium labor market tightness will decrease, while the effect on the wage are in principle ambiguous. The results of an increase in value of c from 0.75 to 1 are shown in Figure 8.

6 Efficiency

Proposition 6.1 (Hosios condition). *Suppose that the matching function is $M(u, v) = Au^\alpha v^{1-\alpha}$. Then the stationary equilibrium if efficient if and only if $\alpha = \phi$.*

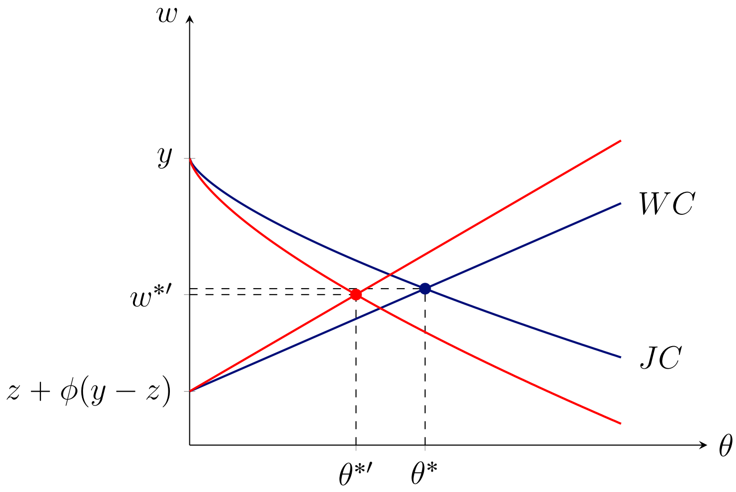

Figure 8: Comparative statics. Effect of increase in c.

Proof. Homework. ■

The sources of inefficiency in this model are:

- Hold-up problem: The vacancy-posting cost is already sunk when worker and firm bargain.
- Search externalities (congestion): Firms are not taking into the account that they lower other firms' hiring probabilities when posting a vacancy.[6]

To get intuition, the following two extremes are interesting to consider:

- If the worker had all bargaining power, we would have $w = y$ and no firm would enter, which is obviously inefficient (hold-up problem! The firm could never recover the sunk entry cost).
- If the firm had all bargaining power, we would have $w = z$. Then there would be too much entry: Firms would post too much, having an externality on the other firms. Under the Hosios condition, the two forces from these extreme scenarios are exactly balanced.

References

Ljungqvist, L. and Sargent, T. J. (2012), *Recursive Macroeconomic Theory*, The MIT Press.

[6] Also workers have a search externality if search intensity is endogenous.