Monetary policy: The New-Keynesian model

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(based on Matthias Kredler's lectures)

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This version: April 28, 2020

Latest version

Abstract

These are notes that I took from the course Macroeconomics II at UC3M, taught by Matthias Kredler during the Spring semester of 2016. Typos and errors are possible, and are my sole responsibility and not that of the instructor.

Contents

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1 Environment

We will follow closely Galí (2008).

- Central idea: Sticky-prices introduce monetary non-neutrality in the short-run.
- Time is discrete and infinite.
- Only shock to TFP: A_t^t (t-th element of the history A^t). We will denote the history up to t as A^t. TFP process is characterized by

$$\ln\left(\frac{A_t}{\bar{A}}\right) = \rho \ln\left(\frac{A_{t-1}}{\bar{A}}\right) + \xi_t^a, \tag{1}$$

  where ξ_t^a is assumed to be zero- mean.
- No capital accumulation.
- Continuum of varieties (different consumption goods) $i \in [0, 1]$:
  - Production function $Y_t(i) = A_t N_t(i)^{1-\alpha}$, where we set $\alpha = 0$.[1]

[1] Check Galí (2008) for $\alpha > 0$.

- The preferences of the consumer exhibit taste-for-variety, i.e. the consumer wants to consume all varieties i.

- Markets:
  - Competitive market for labour. However, the government finances an employment subsidy by which firms pay a nominal wage $(1-\tau)W_t$ and workers receive W_t.
  - Competitive market for bonds.
  - Monopolistic competition in market(s) for consumption goods.

- Sticky prices (Calvo-pricing): only a fraction $(1-\theta)$ of firms $i \in [0,1]$ drawn i.i.d. across time can re-set they price $P_t(i)$ each period.

- Money demand M_t^D: ad-hoc real money-demand depends positively on aggregate consumption, C_t, and negatively on the opportunity cost of holding money. These costs include both shoe leather type of costs but also opportunity costs such as foregone interest on bank accounts, etc. This equation is

$$\frac{M_t^D}{P_t} = \frac{C_t}{I_t^\eta}, \tag{2}$$

  where I_t is the gross nominal interest rate and $\eta > 0$. We assume that a Central Bank controls the money supply $\{M_t^S\}_{t=0}^\infty$.

- The government uses lump-sum transfers T_t to finance the employment subsidy τ.

2 Household's Problem

Taking as given prices of all consumption goods $P_t(i)$, $i \in [0,1]$, bond price[2], Q_t, nominal wage rate, W_t, profits (or dividends), D_t, and transfers, T_t, the household solves

$$\max_{\left\{\{C_t(i)\}_{i\in[0,1]}, C_t, N_t, B_t\right\}_{t=0}^\infty} \quad \mathbb{E}_0\left[\sum_{t=0}^\infty \beta^t u(C_t, N_t)\right],$$

$$\text{s.t.} \quad C_t = \left[\int_0^1 C_t(i)^{\frac{\varepsilon-1}{\varepsilon}}\,\mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}}, \quad \forall t, \quad \varepsilon > 1, \tag{3}$$

$$\int_0^1 P_t(i)C_t(i)\,\mathrm{d}i + Q_tB_t \leqslant B_{t-1} + W_tN_t + D_t + T_t, \quad \forall t, \tag{4}$$

$$B_{-1} = 0 \text{ given,}$$

[2]Note that Q_t is the price paid in t for a bond that gives a safe return of 1 tomorrow.

where N_t are total hours worked and B_t is the quantity of bonds bought. To have a well defined solution, the above sequence of budget constraints is supplemented with the following solvency condition (No-Ponzi condition)

$$\lim_{T\to\infty} \mathbb{E}_t [B_T] \geqslant 0, \quad \forall t.$$

Equation (3) is called Dixit-Stiglitz or CES (Constant Elasticity of Substitution) aggregator. The intuition behind this operator is very close to a CES utility function or production function (i.e. $\left[\alpha c_t^{-\rho} + (1-\alpha) y_t^{-\rho}\right]^{-1/\rho}$). For simplicity, we can think about this as a firm that produces a final good, C_t, using as intermediates all the varieties $C_t(i)$, $i \in [0,1]$.

To make this problem more tractable, we divide it into two sub-problems, an intratemporal problem and an intertemporal problem.

2.1 Sub-problem 1: Intratemporal Problem

At any given period t, the household chooses varieties $C_t(i)$, $\forall i$ in order to minimize the expenditure needed to obtain a given level of the aggregate consumption C_t. That is, taking as given $P_t(i)$, $\forall i$ the household solves

$$\begin{aligned} \min_{\{C_t(i)\}_{i\in[0,1]}} \quad & \int_0^1 P_t(i) C_t(i)\,\mathrm{d}i \\ s.t. \quad C_t &= \left[\int_0^1 C_t(i)^{\frac{\varepsilon-1}{\varepsilon}}\,\mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}}. \end{aligned}$$

The Lagrangean associated to this problem is given by

$$\mathscr{L}\left(\{C_t(i)\}_{i\in[0,1]}, \lambda\right) = -\int_0^1 P_t(i) C_t(i)\,\mathrm{d}i + \lambda\left(\left[\int_0^1 C_t(i)^{\frac{\varepsilon-1}{\varepsilon}}\,\mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}} - C_t\right),$$

with F.O.C.[3]

$$\frac{\partial \mathscr{L}}{\partial C_t(i)} = 0 \Leftrightarrow -P_t(i) + \lambda \frac{\varepsilon}{\varepsilon-1}\left[\int_0^1 C_t(i)^{\frac{\varepsilon-1}{\varepsilon}}\,\mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}-1} \frac{\varepsilon-1}{\varepsilon} C_t(i)^{\frac{\varepsilon-1}{\varepsilon}-1} = 0, \quad \forall i.$$

Working out the previous expression we obtain

$$P_t(i) = \lambda \left[\int_0^1 C_t(i)^{\frac{\varepsilon-1}{\varepsilon}}\,\mathrm{d}i\right]^{\frac{1}{\varepsilon-1}} C_t(i)^{-\frac{1}{\varepsilon}}, \quad \forall i,$$

which can also be rewritten as

$$P_t(i) = \lambda C_t^{\frac{1}{\varepsilon}} C_t(i)^{-\frac{1}{\varepsilon}} \quad \forall i.$$

[3] Assuming interior solution

Solving for $C_t(i)$ yields

$$C_t(i) = \lambda^{\varepsilon} C_t P_t(i)^{-\varepsilon}, \quad \forall i, \tag{5}$$

which tells us how consumption reacts to a change in its own price, i.e., it gives some intuition about the price-elasticity of this good (we will develop this further soon). To find the value of λ we plug this equation into the constraint of the problem, which gives

$$C_t = \left[\int_0^1 \left(\lambda^{\varepsilon} C_t P_t(i)^{-\varepsilon}\right)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}} = \lambda^{\varepsilon} C_t \left[\int_0^1 P_t(i)^{-(\varepsilon-1)} \mathrm{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}}.$$

Finally, clearing for λ yields

$$\lambda = \underbrace{\left[\int_0^1 P_t(i)^{1-\varepsilon} \mathrm{d}i\right]^{\frac{1}{1-\varepsilon}}}_{\equiv P_t\text{: price index}}. \tag{6}$$

The interpretation of the multiplier λ in this constrained-maximization problem is as follows: λ captures the reduction in expenditure under the optimal plan when lowering the required C_t that we want to achieve by one marginal unit. Therefore we can also interpret it as the marginal cost, or price P_t, of C_t. Substituting now (6) in (5) we can obtain

$$C_t^*(i) = \left(\left[\int_0^1 P_t(i)^{1-\varepsilon} \mathrm{d}i\right]^{\frac{1}{1-\varepsilon}}\right)^{\varepsilon} C_t P_t(i)^{-\varepsilon} = P_t^{\varepsilon} C_t P_t(i)^{-\varepsilon} = \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} C_t, \tag{7}$$

which characterizes the demand function for the variety i (once the aggregate demand is known). To obtain the price-elasticity we take logs in the previous equation

$$\ln(C_t(i)) = -\varepsilon \ln(P_t(i)) - [-\varepsilon \ln(P_t)] + \ln(C_t),$$

and then we take the derivative of the optimal consumption with respect to its price obtaining

$$\frac{\mathrm{d}(\ln[C_t(i)])}{\mathrm{d}(\ln[P_t(i)])} = -\varepsilon$$

which is the price elasticity of the demand function. The interpretation is as follows, if the price of consumption good (variety i) rises by 1%, then consumption is reduced in ε% units, for any single price level and any consumption level (this is given by the constant elasticity of substitution).

2.2 Sub-problem 2: Inter-Temporal Problem

Conditional on optimal behaviour of the household when choosing $C_t(i)$, substituting (7) in the first term of (4) we obtain

$$\int_0^1 P_t(i) C_t^*(i) \,\mathrm{d}i = \int_0^1 P_t(i) \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} C_t \,\mathrm{d}i = P_t C_t,$$

i.e. total consumption expenditures can be written as the product of the price index times the quantity index. The first-order conditions are (derive them in your HW!)

$$\frac{W_t}{P_t} = -\frac{u_N(C_t, N_t)}{u_C(C_t, N_t)}, \quad \forall t, \tag{8}$$

$$u_C(C_t, N_t) = \beta \mathbb{E}_t \left[\frac{P_t}{Q_t P_{t+1}} u_C(C_{t+1}, N_{t+1}) \right], \quad \forall t. \tag{9}$$

3 Firm's Problem

- Continuum of firms, $i \in [0, 1]$,
- Production function for variety $i \in [0, 1]$:

$$Y_t(i) = A_t N_t(i).$$

- Calvo pricing:
  - With probability θ, a firm must stick with its old price, i.e. $P_t(i) = P_{t-1}(i)$,
  - With probability $(1 - \theta)$, a firm can change its price, i.e. in general $P_t(i) \neq P_{t-1}(i)$,
- Demand function for each variety $i \in [0, 1]$:

$$Y_t(i) = C_t^*(i) = C_t P_t^{\varepsilon} P_t(i)^{-\varepsilon}$$

- Cost function of the firm. We assume that the firm takes as given the price of labor, $W_t(i)$. Moreover, we assume that a fraction τ of the wage is subsidized.

3.1 Sub-problem: Firm as a Cost Minimizer

We assume that firms are always free to choose how much labor to use each period, but not whether they can adjust their price or not. Hence, let's first consider the problem of optimal labor choice for a given price. Since the firm can freely hire labor each period, we can write this as a static problem. In particular, note that the firm will always choose labor so as to minimize its cost, regardless of its price.

The cost minimization problem of the firm, given the nominal wage W_t, the level of productivity A_t and the desired level of production $Y_t(i)$, is given by

$$\min_{N_t(i)} \quad CF_t(Y_t(i); A_t, W_t) = W_t(1 - \tau) N_t(i)$$
$$\text{s.t.} \quad Y_t(i) \leqslant A_t N_t(i),$$

where τ is an employment subsidy (i.e. the cost of employment is subsidized at rate τ). From the constraint we can easily see that given A_t and W_t, the unique way to produce a given level of $Y_t(i)$, is by hiring

$$N_t(Y_t(i); A_t, W_t) = \frac{Y_t(i)}{A_t} \equiv N_t^*(i), \tag{10}$$

which is the conditional factor demand of $N_t(i)$, which does not depend on the input price W_t or the employment subsidy τ. Substituting this expression in the objective function (or criterion) yields the cost function

$$CF_t(Y_t(i); A_t, W_t) = W_t(1-\tau)N_t^*(i) = W_t(1-\tau)\frac{Y_t(i)}{A_t} = \frac{W_t(1-\tau)}{A_t}Y_t(i).$$

Note that marginal cost of this firm is given by

$$MC_t \equiv \frac{\partial CF_t^*(Y_t(i); A_t, W_t)}{\partial Y_t(i)} = \frac{W_t(1-\tau)}{A_t} = \frac{\widetilde{W}_t}{A_t}, \quad \forall i, \forall t. \tag{11}$$

3.2 Monopolistic competition: Flexible-prices benchmark

Assume that $\theta = 0$, i.e. each firm can set a new price in each period. In this case, the firm does not face a sequential problem[4]. Although we know that the firm only chooses the price or the quantity (in particular, the price, since firms are price-setters) we state the maximization problem with both prices and quantities. Thus the firm's problem reads out as

$$\begin{aligned} \max_{P_t(i), Y_t(i)} \quad & D_t(i) \equiv P_t(i)Y_t(i) - W_t(1-\tau)N_t(i) = Y_t(i)\left[P_t(i) - MC_t(i)\right] \\ \text{s.t.} \quad & Y_t(i) = C_t P_t^{\varepsilon} P_t(i)^{-\varepsilon}, \end{aligned}$$

where $D_t(i)$ is the Profit (or Dividend that the firm pays out to its owners) of Firm i at time t. Substituting the constraint into the maximization problem we can rewrite it as

$$\max_{P_t(i)} \quad D_t(i) = C_t P_t^{\varepsilon} P_t(i)^{-\varepsilon}\left[P_t(i) - MC_t(i)\right],$$

or, equivalently,

$$\max_{P_t(i)} \quad D_t(i) = C_t P_t^{\varepsilon}\left[P_t(i)^{1-\varepsilon} - MC_t(i)P_t(i)^{-\varepsilon}\right].$$

[4]Note that we could as well define the problem with contingent histories, but in the end $P_t(i)$ would only appear in one period and thus the result would be exactly the same as solving the problem period by period.

The F.O.C. is given by[5]

$$\frac{\partial D_t(i)}{\partial P_t(i)} = 0 \iff C_t P_t^{\varepsilon}\left[(1-\varepsilon)P_t(i)^{-\varepsilon} - (-\varepsilon)MC_t(i)P_t(i)^{-\varepsilon-1}\right] = 0$$
$$\iff (1-\varepsilon)P_t(i)^{-\varepsilon} + \varepsilon MC_t(i)P_t(i)^{-\varepsilon-1} = 0$$
$$\iff (\varepsilon-1)P_t(i)^{-\varepsilon} = \varepsilon MC_t(i)P_t(i)^{-\varepsilon-1}.$$

Multiplying both sides of the previous equation by $P_t(i)^{\varepsilon+1}$ and re-araging gives

$$P_t^{*,n}(i) = \frac{\varepsilon}{\varepsilon-1}MC_t(i) = \left[\frac{\varepsilon}{\varepsilon-1} - \frac{\varepsilon-1}{\varepsilon-1} + 1\right]MC_t(i) = \left[1 + \underbrace{\frac{1}{\varepsilon-1}}_{\text{mark-up}}\right]MC_t(i), \quad (12)$$

where $P_t^{*,n}(i)$ stands for the equilibrium price under flexible pricing benchmark. We will refer to this benchmark as the natural equilibrium. Note that the mark-up is decreasing in ε, in fact, as $\varepsilon \to \infty$, we approach the price obtained with perfect competition.

In this flexible price-setting benchmark, firms are symmetric and therefore they all choose the same price. It can be shown that $P_t^{*,n}(i)$ features a constant mark-up over marginal cost for any cost function (under CES demand!).

3.3 Monopolistic competition: Sticky-prices

In this case, we have $\theta > 0$, thus the price that a firm sets today affects future profits (or dividends) to the extent that the price can't be changed during some periods. In other word, in this setting $P_t(i)$ affects not only the present $D_t(i)$ but also future $D_{t+s}(i)$, $s > t$. Let $q_t(i)$ be the current market value of firm i at time t. A firm re-optimizing in period t will choose a price $P_t^*(i)$ that maximizes the current market value of the profits generated while that price remains effective. To define the market value of a firm, note that in equilibrium we must have that the marginal cost of buying firm i at t must be equal to the expected discounted marginal benefit of receiving profits generated this firm from period t onwards i.e.

$$\frac{q_t(i)}{P_t}u_C(C_t, N_t) = \mathbb{E}_t\left[\sum_{s=0}^{\infty}\beta^s \frac{D_{t+s}(i)}{P_{t+s}}u_C(C_{t+s}, N_{t+s})\right].$$

Solving for $q_t(i)$ yields

$$q_t(i) = \mathbb{E}_t\left[\sum_{s=0}^{\infty}\beta^s \frac{u_C(C_{t+s}, N_{t+s})}{u_C(C_t, N_t)}\frac{P_t}{P_{t+s}}D_{t+s}(i)\right].$$

[5] Note that we are safe to assume that the solution will be interior. Since $\varepsilon > 1$ by assumption, then one term is positive and the other negative, thus the function defined by the derivative of $D_t(i)$ will be equal to zero at least once. Suppose that $\varepsilon \in (0, 1)$, then we would have that the derivative would always be positive and then we would not have a well defined solution for this problem.

Let us define the stochastic discount factor[6] (or pricing kernel) as

$$Q_{t,t+s} \equiv \beta^s \frac{u_C(C_{t+s}, N_{t+s})}{u_C(C_t, N_t)} \frac{P_t}{P_{t+s}},$$

thus we can rewrite the previous equation as

$$q_t(i) = \mathbb{E}_t \left[\sum_{s=0}^{\infty} Q_{t,t+s} D_{t+s}(i) \right].$$

3.3.1 Price-Setting

Note that[7]:

- Firms maximize $q_t(i)$ (shareholders value), subject to the sequence of demand constraints
$$Y_{t+s}(i) = C_{t+s}(i) = C_{t+s} P_{t+s}^{\varepsilon} P_{t+s}^{-\varepsilon}(i).$$
- The price $P_t(i)$ only matters in scenarios at $t+s$ if the price set at t it still in place. This event occurs with probability θ^s.

As the price $P_t(i)$ only matters in scenarios at $t+s$ if the price set at t it still in place, the firms solves

$$\begin{aligned} \max_{P_t(i)} \quad q_t(i) &= \mathbb{E}_t \left[\sum_{s=0}^{\infty} \theta^s Q_{t,t+s} C_{t+s} P_{t+s}^{\varepsilon} P_t^{-\varepsilon}(i) \left[P_t(i) - MC_{t+s}(i) \right] \right] \\ &= \mathbb{E}_t \left[\sum_{s=0}^{\infty} \theta^s Q_{t,t+s} C_{t+s} P_{t+s}^{\varepsilon} \left[P_t(i)^{1-\varepsilon} - MC_{t+s}(i) P_t^{-\varepsilon}(i) \right] \right]. \end{aligned}$$

The F.O.C. is given by [8]

$$\frac{\partial q_t(i)}{\partial P_t(i)} = \mathbb{E}_t \left[\sum_{s=0}^{\infty} \theta^s Q_{t,t+s} C_{t+s} P_{t+s}^{\varepsilon} \left[(1-\varepsilon) P_t(i)^{-\varepsilon} - (-\varepsilon) MC_{t+s}(i) P_t(i)^{-\varepsilon-1} \right] \right] = 0,$$

where expanding we obtain

$$-\mathbb{E}_t \left[\sum_{s=0}^{\infty} \theta^s Q_{t,t+s} C_{t+s} P_{t+s}^{\varepsilon} (1-\varepsilon) P_t(i)^{-\varepsilon} \right] = \mathbb{E}_t \left[\sum_{s=0}^{\infty} \theta^s Q_{t,t+s} C_{t+s} P_{t+s}^{\varepsilon} \varepsilon MC_{t+s}(i) P_t(i)^{-\varepsilon-1} \right],$$

or, equivalently

$$(\varepsilon - 1) P_t(i)^{-\varepsilon} \mathbb{E}_t \left[\sum_{s=0}^{\infty} \theta^s Q_{t,t+s} C_{t+s} P_{t+s}^{\varepsilon} \right] = \varepsilon P_t(i)^{-\varepsilon-1} \mathbb{E}_t \left[\sum_{s=0}^{\infty} \theta^s Q_{t,t+s} C_{t+s} P_{t+s}^{\varepsilon} MC_{t+s}(i) \right].$$

[6] Note that it is a random variable, as it depends on the particular history of shocks that take place along time. Therefore we can also define it in terms of histories as $Q_{t,t+s}(A^{t+s})$.

[7] We could also do this with contingent histories, it is a very good exercise.

[8] We again assume interior solution because of the same considerations about ε made before.

Re-arranging we can finally obtain

$$P_t^*(i) = \frac{\varepsilon}{\varepsilon - 1} \frac{\mathbb{E}_t\left[\sum_{s=0}^{\infty} \theta^s Q_{t,t+s} C_{t+s} P_{t+s}^{\varepsilon} MC_{t+s}(i)\right]}{\mathbb{E}_t\left[\sum_{s=0}^{\infty} \theta^s Q_{t,t+s} C_{t+s} P_{t+s}^{\varepsilon}\right]} \tag{13}$$

Note that

- if $MC_{t+s}(i) = MC_t(i)$, $\forall s > t$, then from (13) we have

$$P_t^*(i) = \frac{\varepsilon}{\varepsilon - 1} \frac{\mathbb{E}_t\left[\sum_{s=0}^{\infty} \theta^s Q_{t,t+s} C_{t+s} P_{t+s}^{\varepsilon} MC_t(i)\right]}{\mathbb{E}_t\left[\sum_{s=0}^{\infty} \theta^s Q_{t,t+s} C_{t+s} P_{t+s}^{\varepsilon}\right]} = \frac{\varepsilon}{\varepsilon - 1} MC_t(i),$$

  which is the natural optimal price given by (12).

- future marginal costs $MC_{t+s}(i)$ are weighted by the price-setting probability, the stochastic discount factor, aggregate demand and the aggregate price index.

- (13) defines a time-varying mark-up (it is a forward looking object over MC_{t+s}).

4 Closing the model

4.1 Aggregate price dynamics

In this setting, the price index P_t summarizes all the relevant information about prices at time t. Recall that we define the aggregate price index as

$$P_t = \left[\int_0^1 P_t(i)^{1-\varepsilon} \, \mathrm{d}i\right]^{\frac{1}{1-\varepsilon}}. \tag{14}$$

Moreover, note that we assume that the marginal cost is the same across different firms, therefore when setting a new price at time t, all firms will set the same price. Let $S(t) \subset [0,1]$ represent the set of firms not re-optimizing their posted price at period t, then we can write

$$\begin{aligned} P_t &= \left[\int_{S(t)} \underbrace{P_{t-1}(i)^{1-\varepsilon}}_{\text{price-stickers}} \mathrm{d}i + \underbrace{(1-\theta)(P_t^*)^{1-\varepsilon}}_{\text{price-changers}}\right]^{\frac{1}{1-\varepsilon}} \\ &= \left[\theta P_{t-1}^{1-\varepsilon} + (1-\theta)(P_t^*)^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}, \end{aligned} \tag{15}$$

where the second equality follows from the fact that the distribution of prices among firms not adjusting in period t corresponds to the distribution of effective prices in period $t-1$, though with total mass reduced to θ.

Now we can easily compute the inflation rate. Note that dividing both sides of (15) by P_{t-1} we obtain

$$\Pi_t \equiv \frac{P_t}{P_{t-1}} = \left[\theta + (1-\theta)\left[\frac{P_t^*}{\bar{P}_{t-1}}\right]^{1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}. \tag{16}$$

4.2 Aggregate production function

We still need to derive one more equation, which we will call 'aggregate production function'. Note that from the labor market clearing condition we must have that

$$N_t = \int_0^1 N_t(i)\,\mathrm{d}i,$$

where substituting the production function of each variety i yields

$$N_t = \int_0^1 \frac{Y_t(i)}{A_t}\,\mathrm{d}i = \frac{1}{A_t}\int_0^1 C_t(i)\,\mathrm{d}i = \frac{1}{A_t}\int_0^1 C_t\left[\frac{P_t(i)}{P_t}\right]^{-\varepsilon}\mathrm{d}i = \frac{1}{A_t}Y_t\int_0^1\left[\frac{P_t(i)}{P_t}\right]^{-\varepsilon}\mathrm{d}i,$$

where the second and fourth equalities come from market clearing (i.e. $C_t(i) = Y_t(i)$, $\forall i$ which also implies $C_t = Y_t$), and the third equality follows from equation (7). Re-arranging we can obtain

$$Y_t = \frac{A_t N_t}{\displaystyle\int_0^1 \left[\frac{P_t(i)}{P_t}\right]^{-\varepsilon}\mathrm{d}i}, \tag{17}$$

where

$$d_t \equiv \int_0^1 \left[\frac{P_t(i)}{P_t}\right]^{-\varepsilon}\mathrm{d}i = \begin{cases} = 1 & \text{if} \quad P_t(i) = \text{constant}, \ \forall i. \\ > 1 & \text{if} \quad \text{there is price dispersion.} \end{cases}$$

is a measure of price dispersion.

4.3 Utility function

We will assume that the households' tastes are represented by the utility function

$$u(C_t, N_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi}, \tag{18}$$

where σ and φ are positive parameters.

5 Equilibrium

5.1 Planner's problem

Assuming that the first-order conditions of the planner's problem are both necessary and sufficient for optimalilty (this can be shown using a standard concavity argument), the efficient allocation of this model is characterized by (derive them in your HW!)

$$C_t = C_t(i), \quad \forall i, \forall t \tag{EFF.1}$$

$$A_t = -\frac{u_N(C_t, N_t)}{u_C(C_t, N_t)}, \quad \forall t, \tag{EFF.2}$$

where we note that $C_t = A_t N_t = A_t N_t(i)$ for all i and all t by feasibility.

5.2 Decentralized equilibrium

Definition 5.1 (**Decentralized equilibrium**). *A decentralized equilibrium consists of stochastic sequences for prices* $\{\{P_t(i)\}_{i\in[0,1]}, P_t, P_t^*, W_t, Q_t\}_{t=0}^{\infty}$*, allocations of the household* $\{\{C_t(i)\}_{i\in[0,1]}, C_t, N_t, B_t, M_t^D\}_{t=0}^{\infty}$*, allocations of firms* $\{\{Y_t(i), N_t(i), D_t(i)\}_{i\in[0,1]}\}_{t=0}^{\infty}$ *and government policies* $\{M_t^S, T_t\}_{t=0}^{\infty}$ *such that, for all* t*:*

1. *Given prices, profits* $\{D_t = \int_0^1 D_t(i)\,\mathrm{d}i\}_{t=0}^{\infty}$ *and transfers* $\{T_t\}_{t=0}^{\infty}$*, the allocation of the household* $\{\{C_t(i)\}_{i\in[0,1]}, C_t, N_t, B_t\}_{t=0}^{\infty}$ *solves the Household's problem.*

2. *Given* $\{P_{t+h}, W_{t+h}, C_{t+h}\}_{h>t}$*, the allocation* $\{\{Y_t(i), N_t(i), D_t(i)\}_{i\in[0,1]}\}_{t=0}^{\infty}$ *solves the Firm's problem. Besides, firms that get to choose prices choose* $P_t(i) = P_t^*$*, while the other firms maintain* $P_t(i) = P_{t-1}(i)$*. Labor demand is* $N_t(i) = Y_t(i)/A_t$*.*

3. P_t *is consistent with* $\{P_t(i)\}_{i\in[0,1]}$*, i.e.* (14) *holds.*

4. *Government budget is balanced, i.e.*

$$T_t + \int_0^1 W_t \tau N_t(i)\,\mathrm{d}i = M_t^S - M_{t-1}^S.$$

5. *Markets clear:*

$$\begin{aligned} N_t &= \int_0^1 N_t(i)\,\mathrm{d}i, \\ Y_t(i) &= C_t(i), \quad \forall i, \\ B_t &= 0, \\ M_t^D &= M_t^S. \end{aligned}$$

5.3 Equilibrium equations

The equilibrium equations are given by

$$\ln\left(\frac{A_t}{\bar{A}}\right) = \rho \ln\left(\frac{A_{t-1}}{\bar{A}}\right) + \xi_t^a, \tag{1}$$

$$\frac{M_t}{P_t} = \frac{C_t}{I_t^\eta} \quad \text{(Ad-hoc money demand)}, \tag{2}$$

$$W_t^r = \frac{W_t}{P_t} = -\frac{u_N(C_t, N_t)}{u_C(C_t, N_t)} = \frac{N_t^\varphi}{C_t^{-\sigma}}, \tag{8}$$

$$C_t^{-\sigma} = \beta \mathbb{E}_t\left[\frac{I_t}{\Pi_{t+1}} C_{t+1}^{-\sigma}\right] \quad \text{(combining (9) and (18))}, \tag{19}$$

$$MC_t = \frac{\widetilde{W}_t}{A_t} = \frac{W_t(1-\tau)}{A_t}, \tag{11}$$

$$P_t^* = \frac{\varepsilon}{\varepsilon - 1} \frac{\mathbb{E}_t\left[\sum_{s=0}^{\infty} \theta^s Q_{t,t+s} C_{t+s} P_{t+s}^\varepsilon MC_{t+s}(i)\right]}{\mathbb{E}_t\left[\sum_{s=0}^{\infty} \theta^s Q_{t,t+s} C_{t+s} P_{t+s}^\varepsilon\right]}, \tag{13}$$

$$P_t = \left[\theta P_{t-1}^{1-\varepsilon} + (1-\theta) P_t^{*\,1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}}, \tag{15}$$

$$\Pi_t = \frac{P_t}{P_{t-1}}, \tag{16}$$

$$Y_t = \frac{A_t N_t}{\int_0^1 \left[\frac{P_t(i)}{P_t}\right]^{-\varepsilon} \mathrm{d}i}, \tag{17}$$

$$Y_t = C_t \quad \text{(Market clearing)}, \tag{20}$$

where

$$Q_{t,t+s} \equiv \beta^s \left(\frac{C_{t+s}}{C_t}\right)^{-\sigma} \frac{P_t}{P_{t+s}}.$$

5.4 Deterministic steady state with zero inflation

In a steady state with zero inflation (i.e., $\Pi = 1$) the following hold (derive them in your HW!)

$$\bar{W}^r = \frac{\bar{W}}{\bar{P}} = \frac{\bar{A}}{\widetilde{\mathcal{M}}}, \tag{21}$$

$$\bar{N} = \left(\frac{\bar{A}^{1-\sigma}}{\widetilde{\mathcal{M}}}\right)^{\frac{1}{\varphi+\sigma}}. \tag{22}$$

$$\bar{Y} = \bar{C} = \left(\frac{\bar{A}^{1+\varphi}}{\widetilde{\mathcal{M}}}\right)^{\frac{1}{\varphi+\sigma}}. \tag{23}$$

$$\bar{I} = \frac{1}{\beta} \tag{24}$$

where

$$\widetilde{\mathcal{M}} = \frac{\varepsilon}{\varepsilon - 1}(1 - \tau),$$

5.5 Log-linearized equilibrium equations

In this log-linearization we use the same notation as with Perturbation Methods in the RBC models, i.e. we denote $x_t = \ln X_t$. Recall that this is not the same as $\widetilde{x}_t \equiv \ln X_t - \ln \bar{X}$. The log-linearized versions of the equilibrium equations are (derive them in your HW!)

$$\begin{aligned}
a_t &= \rho a_{t-1} + (\rho - 1) \ln \bar{A} + \xi_t^a, \\
m_t - p_t &= c_t - \eta i_t, && (25) \\
\omega_t - p_t &= \varphi n_t + \sigma c_t, && (26) \\
-\sigma c_t &= i_t + \ln \beta - \mathbb{E}_t \left[\pi_{t+1} \right] - \sigma \mathbb{E}_t \left[c_{t+1} \right], && (27) \\
mc_t &= \omega_t - a_t + \ln(1 - \tau), && (28) \\
p_t^* - p_{t-1} &= (1 - \beta\theta) \sum_{s=0}^{\infty} (\beta\theta)^s \mathbb{E}_t \left[\Theta \widehat{mc}_{t+s}^r + (p_{t+s} - p_{t-1}) \right], && (29) \\
\pi_t &= (1 - \theta)(p_t^* - p_{t-1}), && (30) \\
\pi_t &= p_t - p_{t-1}, && (31) \\
y_t &= a_t + n_t, && (32) \\
y_t &= c_t, && (33)
\end{aligned}$$

where $\Theta = \dfrac{1}{1 - \varepsilon}$ and $\widehat{mc}_{t+s}^r \equiv mc_{t+s}^r - \overline{mc}$, is the log-deviation of the marginal cost from its zero-inflation steady state level.

We can work on (13) to gain some intuition about it. To this end, we will use the fact that

$$\frac{P_{t+s}}{P_{t-1}} = \frac{P_t}{P_{t-1}} \times \frac{P_{t+1}}{P_t} \times \frac{P_{t+2}}{P_{t+1}} \times \cdots \times \frac{P_{t+s}}{P_{t+s-1}} = \Pi_t \times \Pi_{t+1} \times \cdots \times \Pi_{t+s} = \prod_{k=0}^{s} \Pi_{t+k},$$

therefore

$$\ln \frac{P_{t+s}}{P_{t-1}} = \ln \prod_{k=0}^{s} \Pi_{t+k} \quad \Longleftrightarrow \quad p_{t+s} - p_{t-1} = \sum_{k=0}^{s} \pi_{t+k}.$$

Then, from (29) we have

$$
\begin{aligned}
p_t^* - p_{t-1} &= (1-\beta\theta)\sum_{s=0}^{\infty}(\beta\theta)^s \mathbb{E}_t\left[\Theta\widehat{mc}_{t+s}^r + (p_{t+s} - p_{t-1})\right] \\
&= (1-\beta\theta)\sum_{s=0}^{\infty}(\beta\theta)^s \mathbb{E}_t\left[\Theta\widehat{mc}_{t+s}^r\right] + (1-\beta\theta)\sum_{s=0}^{\infty}(\beta\theta)^s \mathbb{E}_t\left[\sum_{k=0}^{s}\pi_{t+k}\right] \\
&= (1-\beta\theta)\sum_{s=0}^{\infty}\Theta(\beta\theta)^s \mathbb{E}_t\left[\widehat{mc}_{t+s}^r\right] + \sum_{s=0}^{\infty}(\beta\theta)^s \mathbb{E}_t\left[\pi_{t+s}\right],
\end{aligned}
\tag{34}
$$

where in the last equality we use the fact that

$$
(1-\beta\theta)\sum_{s=0}^{\infty}(\beta\theta)^s \mathbb{E}_t\left[\sum_{k=0}^{s}\pi_{t+k}\right] = \sum_{s=0}^{\infty}(\beta\theta)^s\,\mathbb{E}_t\left[\pi_{t+s}\right].
$$

Note that the above discounted sum in (34) can be rewritten as

$$
\begin{aligned}
p_t^* - p_{t-1} &= (1-\beta\theta)\Theta\widehat{mc}_t^r + \pi_t + \cdots \\
&\cdots + \beta\theta\left[(1-\beta\theta)\sum_{s=0}^{\infty}\Theta(\beta\theta)^s \mathbb{E}_t\left[\widehat{mc}_{t+1+s}^r\right] + \sum_{s=0}^{\infty}(\beta\theta)^s \mathbb{E}_t\left[\pi_{t+1+s}\right]\right],
\end{aligned}
$$

where by the law of iterated expectations we can write

$$
p_t^* - p_{t-1} = (1-\beta\theta)\Theta\widehat{mc}_t^r + \pi_t + \beta\theta\mathbb{E}_t\left[p_{t+1}^* - p_t\right].
$$

Now, using (30) we obtain

$$
\frac{\pi_t}{1-\theta} = (1-\beta\theta)\Theta\widehat{mc}_t^r + \pi_t + \beta\theta\mathbb{E}_t\left[\frac{\pi_{t+1}}{1-\theta}\right],
$$

which can be rewritten as

$$
\pi_t = \underbrace{\frac{(1-\theta)(1-\beta\theta)\Theta}{\theta}}_{\equiv\lambda}\widehat{mc}_t^r + \beta\mathbb{E}_t\left[\pi_{t+1}\right]. \tag{35}
$$

Solving (35) forward yields

$$
\begin{aligned}
\pi_t &= \lambda\widehat{mc}_t^r + \beta\mathbb{E}_t\left[\pi_{t+1}\right] \\
&= \lambda\widehat{mc}_t^r + \beta\mathbb{E}_t\left[\lambda\widehat{mc}_{t+1}^r + \beta\mathbb{E}_t\left[\pi_{t+2}\right]\right] = \lambda\widehat{mc}_t^r + \lambda\beta\mathbb{E}_t\left[\widehat{mc}_{t+1}^r\right] + \beta^2\mathbb{E}_t\left[\pi_{t+2}\right] \\
&= \lambda\widehat{mc}_t^r + \lambda\beta\mathbb{E}_t\left[\widehat{mc}_{t+1}^r\right] + \beta^2\mathbb{E}_t\left[\lambda\widehat{mc}_{t+2}^r + \beta\mathbb{E}_t\left[\pi_{t+3}\right]\right] \\
&= \cdots \\
&= \lambda\sum_{s=0}^{\infty}\beta^s\mathbb{E}_t\left[\widehat{mc}_{t+s}^r\right].
\end{aligned}
$$

It is worth emphasizing here that the mechanism underlying fluctuations in the aggregate price level and inflation as laid out above has little in common with the mechanism at work in the classical monetary economy. Thus, in the present model, inflation results from the aggregate consequences of purposeful price-setting decisions by firms, which adjust their prices in light of current and anticipated cost conditions. By contrast, in the classical monetary economy, inflation is a consequence of the changes in the aggregate price level that, given the monetary policy rule in place, are required in order to support an equilibrium allocation that is independent of the evolution of nominal variables, with no account given of the mechanism (other than an invisible hand) that will bring about those price level changes.

5.6 New Keynesian Phillips Curve

To obtain the New Keynesian Phillips Curve, we first need to derive a relationship between output y_t (which we know that in equilibrium will be equal to c_t by market clearing) and the marginal cost, mc_t^r. To this end, note that

$$\begin{aligned} mc_t^r &= mc_t - p_t \\ &= \omega_t - a_t + \underbrace{\ln(1-\tau)}_{\tilde{\tau}} - p_t \\ &= \varphi n_t + \sigma c_t + \tilde{\tau} - a_t \\ &= \varphi(y_t - a_t) + \sigma y_t - a_t + \tilde{\tau} \\ &= (\varphi + \sigma) y_t - (1+\varphi) a_t + \tilde{\tau}, \end{aligned} \tag{36}$$

where the second equality follows from substituting (28), the third from substituting (26) and the fourth from substituting (32) and (33). We want to write this equation as output gap, which in a New-Keynesian model is obtained as the deviation of output from its level under flexible prices. Under flexible prices ('natural' level), the real marginal cost $mc_t^{r,n}$ is constant[9] (and equal to its steady-state level) and is given by

$$mc_t^{r,n} = (\varphi + \sigma) y_t^n - (1+\varphi) a_t + \tilde{\tau}. \tag{37}$$

Note that now we denote y_t^n with a superscript n that denotes 'natural' output level under flexible prices. Now, subtracting (36) from (37) we obtain

$$\begin{aligned} mc_t^r - mc_t^{r,n} &= (\varphi + \sigma) y_t - (1+\varphi) a_t + \tilde{\tau} - [(\varphi+\sigma) y_t^n - (1+\varphi) a_t + \tilde{\tau}] \\ &= (\varphi+\sigma)\left(y_t - y_t^n\right), \end{aligned}$$

[9] All firms set the same price, see your HW!

where, again, we define $\widehat{mc}_t^r \equiv mc_t^r - mc_t^{r,n} = mc_t^r - \overline{mc}$. Finally, to obtain the New Keynesian Phillips Curve, we substitute this expression in (35) obtaining

$$\pi_t = \lambda(\varphi + \sigma)\left(y_t - y_t^n\right) + \beta \mathbb{E}_t\left[\pi_{t+1}\right],$$

where denoting by $\widetilde{y}_t = y_t - y_t^n$ the deviation of output from its 'natural' level, and defining

$$\kappa \equiv \lambda(\varphi + \sigma),$$

we obtain

$$\pi_t = \beta \mathbb{E}_t\left[\pi_{t+1}\right] + \kappa \widetilde{y}_t, \tag{38}$$

which is called the New-Keynesian Phillips curve.

Interpretation: if we have a low price level then the real wage will be high as the nominal wage is fixed. Therefore, when agents make their labour supply decision they will decide to work more hours ('opportunity' cost of leisure increases) and thus, as more hours are worked, production will increase. Taking into account that the real marginal cost will also be higher, then the firms will tend to set higher prices when they are touched by the 'Calvo fairy', and this would introduce an inflationary trend in the economy, thus in the end, inflation will rise.

5.7 Dynamic IS Curve

To obtain the Dynamic IS Curve we proceed as follows. First we start from (27), where we set $\ln \beta = -\rho$ and use the fact that in equilibrium $c_t = y_t$, thus we can write

$$y_t = -\frac{1}{\sigma}\left(-\rho + i_t - \mathbb{E}_t\left[\pi_{t+1}\right]\right) + \mathbb{E}_t\left[y_{t+1}\right],$$

then subtracting the natural output level y_t^n we have

$$y_t - y_t^n = -\frac{1}{\sigma}\left(-\rho + i_t - \mathbb{E}_t\left[\pi_{t+1}\right]\right) + \mathbb{E}_t\left[y_{t+1}\right] - y_t^n,$$

since y_t^n is known at t we can include it inside the expectation term, where we sum and subtract y_{t+1}^n keeping the equality unchanged. Furthermore, as before we define $\widetilde{y}_t \equiv y_t - y_t^n$ obtaining

$$\widetilde{y}_t = -\frac{1}{\sigma}\left(-\rho + i_t - \mathbb{E}_t\left[\pi_{t+1}\right]\right) + \mathbb{E}_t\left[y_{t+1} - y_{t+1}^n + y_{t+1}^n - y_t^n\right],$$

where we can rewrite $\widetilde{y}_{t+1} = y_{t+1} - y_{t+1}^n$ and define $\Delta y_{t+1}^n \equiv y_{t+1}^n - y_t^n$ as the 'natural' change in output, obtaining

$$\begin{aligned}
\widetilde{y}_t &= -\frac{1}{\sigma}\left(-\rho + i_t - \mathbb{E}_t\left[\pi_{t+1}\right]\right) + \mathbb{E}_t\left[\widetilde{y}_{t+1} + \Delta y_{t+1}^n\right] \\
&= -\frac{1}{\sigma}\left(-\rho + i_t - \mathbb{E}_t\left[\pi_{t+1}\right]\right) + \mathbb{E}_t\left[\widetilde{y}_{t+1}\right] + \mathbb{E}_t\left[\Delta y_{t+1}^n\right] \\
&= -\frac{1}{\sigma}\left(-\rho + i_t - \mathbb{E}_t\left[\pi_{t+1}\right] - \sigma\mathbb{E}_t\left[\Delta y_{t+1}^n\right]\right) + \mathbb{E}_t\left[\widetilde{y}_{t+1}\right] \\
&= -\frac{1}{\sigma}\left[i_t - \mathbb{E}_t\left[\pi_{t+1}\right] - \left(\rho + \sigma\mathbb{E}_t\left[\Delta y_{t+1}^n\right]\right)\right] + \mathbb{E}_t\left[\widetilde{y}_{t+1}\right].
\end{aligned}$$

The natural interest rate, r_t^n, is given by[10]

$$r_t^n \equiv \rho + \sigma\mathbb{E}_t\left[\Delta y_{t+1}^n\right] = \rho + \sigma\frac{1+\varphi}{\sigma+\varphi}\mathbb{E}_t\left[\Delta a_{t+1}\right],$$

thus we can rewrite the previous expression as

$$\widetilde{y}_t = -\frac{1}{\sigma}\left(i_t - \mathbb{E}_t\left[\pi_{t+1}\right] - r_t^n\right) + \mathbb{E}_t\left[\widetilde{y}_{t+1}\right], \tag{39}$$

which is called Dynamic IS curve. Note that the term $i_t - \mathbb{E}_t\left[\pi_{t+1}\right] - r_t^n$ is the deviation of the real interest rate from its natural level.

5.8 (Ad-hoc) LM Curve

The LM Curve is given by the log-linearized real money demand (real balances), given by

$$m_t - p_t = y_t - \eta i_t. \tag{40}$$

6 Determination of dynamics

The dynamics of our model are summarized in the equations

$$\begin{aligned}
\pi_t &= \beta\mathbb{E}_t\left[\pi_{t+1}\right] + \kappa\widetilde{y}_t, && (38)\\
\widetilde{y}_t &= -\frac{1}{\sigma}\left(i_t - \mathbb{E}_t\left[\pi_{t+1}\right] - r_t^n\right) + \mathbb{E}_t\left[\widetilde{y}_{t+1}\right], && (39)\\
m_t - p_t &= y_t - \eta i_t. && (40)
\end{aligned}$$

Note that in this system of equations, our endogenous objects (and the ones we want to look for) are $\widetilde{y}_t$, π_t and i_t. To solve this model we have to find stochastic sequences[11] $\{\widetilde{y}_t, \pi_t, i_t\}_{t=0}^{\infty}$ that solve (38), (39) and (40), where $\{m_t, r_t^n\}_{t=0}^{\infty}$ are exogenous sequences.

[10] See your homework!

[11] In fact, we should look for stochastic sequences contingent on histories, i.e. $\left\{\{\widetilde{y}_t(A^t), \pi_t(A^t), i_t(A^t)\}_{t=0}^{\infty}\right\}_{A^t\in\mathcal{A}^t}$

Two ways of thinking about monetary policy:

1. Fix a process for m_t (approach followed by Galí (2008)). For example, we suppose that money balances are defined by the following AR(1) process

$$\Delta m_t = \rho_m \Delta m_{t-1} + \varepsilon_t^m.$$

   With this assumption, we can find that there is always a stable solution and a unique equilibrium to this problem.

2. Act as if the Central Bank can choose $\{i_t\}_{t=0}^{\infty}$ (which, in fact, is not a very unrealistic assumption). In this case i_t becomes an exogenous object. Then we can

   (a) solve for sequences $\{\widetilde{y}_t, \pi_t\}_{t=0}^{\infty}$ given a rule for $\{i_t\}_{t=0}^{\infty}$ that solve (38) and (39), and

   (b) back out the rule for $\{m_t\}_{t=0}^{\infty}$ from equation (40).

6.1 Monetary policy analysis

We will follow the second way of thinking about monetary policy. In particular, we will assume that the Central Bank sets the following interest rate rule

$$i_t = \rho + \phi_y \widetilde{y}_t + \phi_\pi \pi_t + \nu_t, \tag{41}$$

i.e., the nominal interest rate fluctuates around ρ (the steady state level given by the efficient rule), where ν_t is defined as

$$\nu_t = \rho_\nu \nu_{t-1} + \varepsilon_t^\nu, \tag{42}$$

and we assume that if follows an AR(1) zero mean process, i.e. $\mathbb{E}[\nu_t] = \mathbb{E}[\varepsilon_t^\nu] = 0$.

Our goal with this monetary policy rule is to solve for $\{\widetilde{y}_t, \pi_t\}_{t=0}^{\infty}$ as a function of $\mathbb{E}_t[\pi_{t+1}]$ and $\mathbb{E}_t[\widetilde{y}_{t+1}]$ and the shocks. In particular we are looking for a solution in the same linear form as we did in the RBC model, i.e.

$$\begin{bmatrix} \widetilde{y}_t \\ \widetilde{\pi}_t \end{bmatrix} = \widetilde{A}\mathbb{E}_t \left\{ \begin{bmatrix} \widetilde{y}_{t+1} \\ \widetilde{\pi}_{t+1} \end{bmatrix} \right\} + \widetilde{B}\varepsilon_t.$$

Combining (38) and (39) with (41) and (42) yields (derive in your HW!)

$$\begin{bmatrix} \widetilde{y}_t \\ \pi_t \end{bmatrix} = \underbrace{\Omega \begin{bmatrix} \sigma & 1 - \beta\phi_\pi \\ \sigma\kappa & \kappa + \beta(\sigma + \phi_y) \end{bmatrix}}_{\equiv \widetilde{A}} \mathbb{E}_t \begin{bmatrix} \widetilde{y}_{t+1} \\ \pi_{t+1} \end{bmatrix} + \underbrace{\Omega \begin{bmatrix} 1 \\ \kappa \end{bmatrix}}_{\equiv \widetilde{B}} (\widehat{r}_t^n - \nu_t), \tag{EQM}$$

where

- $\widehat{r}_t^n$ is the deviation of the real interest rate from its steady state (which turns out to be proportional to the productivity shock) and,

- ν_t is the monetary policy shock.

Note that we can rewrite (EQM) in Blanchard-Khan form as

$$\mathbb{E}_t \begin{bmatrix} \widetilde{y}_{t+1} \\ \pi_{t+1} \end{bmatrix} = \widetilde{A}^{-1} \begin{bmatrix} \widetilde{y}_t \\ \pi_t \end{bmatrix} + \widetilde{A}^{-1} \widetilde{B} (\widehat{r}_t^n - \nu_t). \tag{43}$$

Given that both output gap and inflation are non-predetermined variables, the solution to (43) is locally unique, if and only if, $\widetilde{A}^{-1}$ has both eigenvalues strictly outside the unit circle (i.e. > 1 in absolute value), or, equivalently, if and only if, $\widetilde{A}$ has both eigenvalues strictly inside the unit circle (i.e. < 1 in absolute value).

Under the assumption of non-negative coefficients ϕ_π, ϕ_y, with some algebra it can be shown that a necessary and sufficient condition for uniqueness is given by[12]

$$\kappa(\phi_\pi - 1) + (1 - \beta)\phi_y > 0 \tag{STAB}$$

which is assumed to hold, unless stated otherwise. The interpretation[13] of this equation is as follows: the Central Bank should react strongly (or in other terms, more than one-to-one) against inflation and the output gap. A rough intuition:

- If π_t increases, then the Central Bank should increase i_t by more than one-to-one (contractive monetary policy).

- Besides, if $\widetilde{y}_t$ increases, then the Central Bank should also increase i_t by more than one-to-one (contractive monetary policy).

Otherwise, if (STAB) does not hold, the we have infinitely many solutions to the sequences $\{\widetilde{y}_t, \pi_t\}_{t=0}^{\infty}$ that solve (EQM) and that are locally stable.

To obtain a closed form solution to the system of equations (EQM), we will use the method of undetermined coefficients. To this end, we guess that the solution of this system of equations is linear and takes the following form

$$\widetilde{y}_t = \psi_{yr} \widehat{r}_t^n + \psi_{y\nu} \nu_t, \tag{44}$$

$$\pi_t = \psi_{\pi r} \widehat{r}_t^n + \psi_{\pi\nu} \nu_t. \tag{45}$$

[12] Galí (2008) refers to Bullard and Mitra (2002) for a proof. Reference: Bullard, James, and Kaushik Mitra (2002): "Learning About Monetary Policy Rules," Journal of Monetary Economics 49, no. 6, 1105–1130.

[13] A full economic interpretation to the previous can be found in (Galí, 2008, Chapter 4)

Note that (EQM) has to hold for any combination of $(\widehat{r}_t^n, \nu_t)$. Focusing on the response to unexpected monetary shocks, we obtain (derive in your HW!)

$$\psi_{y\nu} = -(1-\beta\rho_\nu)\Lambda_\nu, \tag{46}$$

$$\psi_{\pi\nu} = -\kappa\Lambda_\nu, \tag{47}$$

where

$$\Lambda_\nu = \frac{1}{(1-\beta\rho_\nu)[\sigma(1-\rho_\nu)+\phi_y]+\kappa(\phi_\pi-\rho_\nu)}, \tag{48}$$

and it can be shown that $\Lambda_\nu > 0$ as long as (STAB) is satisfied. Finally, coming back to our guesses (44) (45), if $\widehat{r}_t^n = 0$ we can write

$$\widetilde{y}_t = -(1-\beta\rho_\nu)\Lambda_\nu\nu_t$$
$$\pi_t = -\kappa\Lambda_\nu\nu_t$$

In these two equations we can note the following:

- Short-run effect of unexpected monetary policy shock: If $\nu_t > 0$ (higher nominal rate than expected) then a tighter monetary policy is being undertaken, and $\widetilde{y}_t < 0$, i.e., as a consequence we obtain a negative output gap (lower output). Furthermore, we also obtain low inflation $\pi_t < 0$ ($\cong$ deflation).
- Long-run effect of aggressive policy:
  - If ϕ_π increases, then Λ_ν decreases, and therefore both $\widetilde{y}_t$ and π_t decrease. Note that this is a deep parameter of the monetary policy, and therefore a change in this parameter modifies the response to shocks in ν_t. In particular, a more aggressive response to inflation brings down volatility of output gap and inflation.
  - If ϕ_y increases, then Λ_ν decreases, and therefore both $\widetilde{y}_t$ and π_t decrease. Therefore we obtain the same result as before.
  - The parameters ϕ_π and ϕ_y are called long term parameters.

References

Galí, J. (2008), *Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Framework*, Princeton University Press.