Solving dynamic stochastic general equilibrium (DSGE) models with an application to the real-business-cycle (RBC) model

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(based on Matthias Kredler's lectures)

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Abstract

These are notes that I took from the course Macroeconomics II at UC3M, taught by Matthias Kredler during the Spring semester of 2016. Typos and errors are possible, and are my sole responsibility and not that of the instructor.

Contents

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1 Global Methods

(a) *Value function iteration or policy function iteration:*

Iterate on a grid $\{x_k\}_{k=1}^K$ either

(i) Value function:

$$V_{n+1}(x) = \max_{x' \in \Gamma(x)} \{F(x, x') + \beta V_n(x')\}, \tag{BE}$$

where $V_n \longrightarrow V^*$ as $n \to \infty$.

(ii) Policy function (using first-order conditions):

$$F_2(x, x'^*) + \beta V_n(x'^*) = 0,$$

by the envelope theorem

$$\underbrace{-F_2(x, x'^*)}_{\substack{\text{Marginal loss}\\ \text{today}}} = \underbrace{\beta F_1(x'^*, x''^*)}_{\substack{\text{Marginal benefit}\\ \text{of saving}}},$$

which can be rewritten in terms of the optimal policy function as

$$-F_2(x, \underbrace{g_{n+1}(x)}_{x'}) = \beta F_1(\underbrace{g_{n+1}(x)}_{x'}, \overbrace{g_n(\underbrace{g_{n+1}(x)}_{x'})}^{\substack{x'' \text{ according}\\ \text{to policy } n}}),$$

where $g_n \longrightarrow g^*$ as $n \to \infty$.

(b) *Projection methods:*

Approximate the policy function or the value function, using a basis of the function space, $\{b_j(x)\}_{j=1}^J$ (e.g. polynomials, $b_j(x) = x^j$).

$$V_n(x) = \sum_{j=1}^{J} \alpha_j^{(n)} b_j(x) \implies \text{Vector } \underset{(J\times 1)}{\alpha^n},$$

such that it characterizes completely V_n. Again, we would usually iterate on value or policy functions: Given α_n, find α_{n+1} and wait until α_n settles down. The least-used method in practice.

2 Local Methods

Exploit the idea that, as long as we are close to ('in a neighbourhood of') the steady state of a model, we may approximate the behaviour of the model in that neighbourhood. We call this perturbation methods, and consist on approximating the economy's behaviour around the (deterministic) steady state.

This is the fastest method to solve a model... But not always good in capturing global behaviour (i.e. far away from the steady state of the economy).

2.1 Perturbation: Log-linearization of models

The procedure we follow is called log-linearization, and consists on working with log-deviations from the (deterministic) steady state of the model. To show the procedure, we will first set up the notation, and then we will explain the approach in a general model. Afterwards, we will apply it to a bare-bones real-business-cycle (RBC) model .

2.1.1 Notation

- X_t: aggregate variable (e.g. GDP in €).
- $\bar{X}$: steady state value of the variable X_t (this is obtained when all the shocks are equal to $0, \forall t$, i.e. solving for the deterministic model).
- $x_t \equiv \ln(X_t)$.
- Log-deviations:

$$\tilde{x}_t \equiv \ln\left(\frac{X_t}{\bar{X}}\right) = \underbrace{\ln(X_t) - \ln(\bar{X})}_{\substack{\%\text{ deviation}\\ \text{from steady state}}}.$$

The First-order Taylor approximation of $g(X_t) \equiv \ln\left(X_t/\bar{X}\right)$ around $X_t = \bar{X}$ is given by

$$\begin{aligned}
\tilde{x}_t = g(X_t) &\cong g(\bar{X}) + g'(X_t)|_{X_t=\bar{X}} \left[X_t - \bar{X}\right] \\
&\cong \ln\left(\frac{\bar{X}}{\bar{X}}\right) + \left.\frac{1}{X_t}\right|_{X_t=\bar{X}} \left[X_t - \bar{X}\right] \\
&\cong \frac{X_t - \bar{X}}{\bar{X}}.
\end{aligned}$$

Then $\tilde{x}_t$ is the percentage deviation of X_t from the steady state. [1].

2.2 General Model

- State variables:
  - S_t: $n_S \times 1$ vector of endogenous state variables (e.g. K_t).
  - Z_t: $n_Z \times 1$ vector of exogenous state variables (e.g. A_t).
- Control variables:
  - X_t: $n_X \times 1$ vector of non-states (e.g. C_t).
  - S_{t+1}: $n_S \times 1$ (e.g. K_{t+1}).

2.2.1 First Step: Find the equations that characterize the equilibrium (optimality, feasibility,...)

- n_Z exogenous equations (to be seen later, equivalent to (Eq.0)).
- System of $n_S + n_X$ equations characterizing the solutions (to be seen later, equivalent to (Eq.1), (Eq.2)).
- Note that the number of equations must be equal to that of unknowns.

2.2.2 Second Step: Find the deterministic steady state

- Solve $n_S + n_X$ equilibrium equations for $\left(\underset{n_S\times 1}{\bar{S}}, \underset{n_X\times 1}{\bar{X}}\right)$, setting all shock to zero $\forall t$.

[1] E.g. $\tilde{x}_t = 0.02$ means that the variable X_t is 2% away from it's steady state value $\bar{X}$.

2.2.3 Third Step: Log-linearize the equilibrium equations

(a) First method:

1. Write the equilibrium equations in log deviations, i.e. for each component i of S write
$$S_t^{(i)} = \bar{S}^{(i)} e^{\tilde{s}_t^{(i)}} \implies \ln(S_t^{(i)}) = \ln(\bar{S}^{(i)}) + \tilde{s}_t^{(i)}.$$
Do the same for each component i of X and Z, i.e.,
$$X_t^{(i)} = \bar{X}^{(i)} e^{\tilde{x}_t^{(i)}},$$
$$Z_t^{(i)} = \bar{Z}^{(i)} e^{\tilde{z}_t^{(i)}}.$$

2. Perform a 1st-order Taylor approximation of the equilibrium equations in the variables $\tilde{s}_t^{(i)}, \tilde{x}_t^{(i)}, \tilde{z}_t^{(i)}$ (and their leads and lags) around $\tilde{s}_t^{(i)} = 0, \tilde{x}_t^{(i)} = 0, \tilde{z}_t^{(i)} = 0$ (which is in fact the steady state of the model).

(b) Second method:[2]

1. First take logs of equilibrium conditions.
2. Expand in $\tilde{s}_t^{(i)}, \tilde{x}_t^{(i)}, \tilde{z}_t^{(i)}$.

2.2.4 Fourth Step: Solve for policy

We will use the method of undetermined coefficients. In particular, we will guess a linear form (in terms of the state variables) for the policy function:

$$\underset{(n_X+n_S)\times 1}{\begin{bmatrix} \tilde{X}_t \\ \tilde{S}_{t+1} \end{bmatrix}} = \underset{(n_X+n_S)\times(n_S+n_Z)}{H} \underset{(n_S+n_Z)\times 1}{\begin{bmatrix} \tilde{S}_t \\ \tilde{Z}_t \end{bmatrix}},$$

where

$$H = \begin{bmatrix} \underset{(n_X \times n_S)}{H_{XS}} & \underset{(n_X \times n_Z)}{H_{XZ}} \\ \underset{(n_S \times n_S)}{H_{SS}} & \underset{(n_S \times n_Z)}{H_{SZ}} \end{bmatrix}.$$

Thus we can write

$$\begin{bmatrix} \tilde{X}_t \\ \tilde{S}_{t+1} \end{bmatrix} = \begin{bmatrix} H_{XS} & H_{XZ} \\ H_{SS} & H_{SZ} \end{bmatrix} \begin{bmatrix} \tilde{S}_t \\ \tilde{Z}_t \end{bmatrix}. \tag{H}$$

Consider the log-linearized equilibrium equations given by

$$\underset{(n_X+n_S)\times(n_X+n_S)}{A_0} \underset{(n_X+n_S)\times 1}{\begin{bmatrix} \tilde{X}_t \\ \tilde{S}_t \end{bmatrix}} = \underset{(n_X+n_S)\times(n_X+n_S)}{A_1} \mathbb{E}_t \left\{ \underset{(n_X+n_S)\times 1}{\begin{bmatrix} \tilde{X}_{t+1} \\ \tilde{S}_{t+1} \end{bmatrix}} \right\} + \underset{[(n_X+n_S)\times n_Z]}{B_0} \underset{(n_Z\times 1)}{\tilde{Z}_t}, \tag{A}$$

[2] Homework: Methods 1 and 2 must give exactly the same result.

where the coefficients of the matrices A_0, A_1 and B_0 are a function of model parameters and steady state values (e.g. $\bar{K}$, $\bar{C}$, ...), and

$$\underset{n_Z\times 1}{\tilde{Z}_{t+1}} = \underset{(n_Z\times n_Z)}{A_z}\underset{(n_Z\times 1)}{\tilde{Z}_t} + \underset{n_Z\times 1}{\varepsilon_{t+1}}, \qquad \mathbb{E}\left[\varepsilon_{t+1}\right] = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{Z}$$

Denoting by I_{nS} the $nS \times nS$ identity matrix, and substituting (H) in (A) we have

$$A_0 \begin{bmatrix} H_{XS} & H_{XZ} \\ I_{n_S} & 0 \end{bmatrix} \begin{bmatrix} \tilde{S}_t \\ \tilde{Z}_t \end{bmatrix} = A_1 \mathbb{E}_t \left\{ \begin{bmatrix} H_{XS} & H_{XZ} \\ I_{n_S} & 0 \end{bmatrix} \begin{bmatrix} \tilde{S}_{t+1} \\ \tilde{Z}_{t+1} \end{bmatrix} \right\} + B_0 \tilde{Z}_t.$$

As the matrix H is deterministic, we can express the previous equation as

$$A_0 \begin{bmatrix} H_{XS} & H_{XZ} \\ I_{n_S} & 0 \end{bmatrix} \begin{bmatrix} \tilde{S}_t \\ \tilde{Z}_t \end{bmatrix} = A_1 \begin{bmatrix} H_{XS} & H_{XZ} \\ I_{n_S} & 0 \end{bmatrix} \mathbb{E}_t \left\{ \begin{bmatrix} \tilde{S}_{t+1} \\ \tilde{Z}_{t+1} \end{bmatrix} \right\} + B_0 \tilde{Z}_t. \tag{1}$$

From (H) and (Z) the expectation term can be rewritten as

$$\begin{aligned} \mathbb{E}_t \left\{ \begin{bmatrix} \tilde{S}_{t+1} \\ \tilde{Z}_{t+1} \end{bmatrix} \right\} &= \mathbb{E}_t \left\{ \begin{bmatrix} H_{SS} & H_{SZ} \\ 0 & A_z \end{bmatrix} \begin{bmatrix} \tilde{S}_t \\ \tilde{Z}_t \end{bmatrix} + \begin{bmatrix} 0 \\ I_{n_Z} \end{bmatrix} \varepsilon_{t+1} \right\} \\ &= \begin{bmatrix} H_{SS} & H_{SZ} \\ 0 & A_z \end{bmatrix} \begin{bmatrix} \tilde{S}_t \\ \tilde{Z}_t \end{bmatrix}, \end{aligned} \tag{2}$$

where I_{nZ} denotes the $nZ \times nZ$ identity matrix and where the last equality follows from the matrix being deterministic and because the values of $\tilde{S}_t$ and $\tilde{Z}_t$ are known in t, which allows to get rid of the expectation. Substituting (2) in (1) gives

$$A_0 \begin{bmatrix} H_{XS} & H_{XZ} \\ I_{n_S} & 0 \end{bmatrix} \begin{bmatrix} \tilde{S}_t \\ \tilde{Z}_t \end{bmatrix} = A_1 \begin{bmatrix} H_{XS} & H_{XZ} \end{bmatrix} \begin{bmatrix} H_{SS} & H_{SZ} \\ 0 & A_z \end{bmatrix} \begin{bmatrix} \tilde{S}_t \\ \tilde{Z}_t \end{bmatrix} + B_0 \tilde{Z}_t. \tag{3}$$

Rewrite the last term of the previous equation as

$$B_0 \tilde{Z}_t = \begin{bmatrix} 0 & \underset{n_S\times n_Z}{B_{0S}} \\ 0 & \underset{n_Z\times n_Z}{B_{0Z}} \end{bmatrix} \begin{bmatrix} \tilde{S}_t \\ \tilde{Z}_t \end{bmatrix},$$

then we can rewrite (3) as

$$\underbrace{\left\{ A_0 \begin{bmatrix} H_{XS} & H_{XZ} \\ I_{n_S} & 0 \end{bmatrix} - A_1 \begin{bmatrix} H_{XS} & H_{XZ} \\ I_{n_S} & 0 \end{bmatrix} \begin{bmatrix} H_{SS} & H_{SZ} \\ 0 & A_z \end{bmatrix} - \begin{bmatrix} 0 & B_{0S} \\ 0 & B_{0Z} \end{bmatrix} \right\}}_{=K} \begin{bmatrix} \tilde{S}_t \\ \tilde{Z}_t \end{bmatrix} = 0.$$

This equation has to hold for all $\tilde{S}_t$ and for all $\tilde{Z}_t$. Thus we have that the matrix K must be a matrix with all the elements equal to zero. Note that the dimension of this matrix is the same of the matrix H, i.e. $(n_X + n_S) \times (n_S + n_Z)$. This is the same number of equations that we have to solve in order to get our solution.

In general, we can have multiple solutions to our problem as we have matrix products $H_{XS}H_{SS}$ and $H_{XS}H_{SZ}$, thus the system of equations is (usually) non-linear. To check which of this solutions are valid for our purposes we must check the eigenvalues of the matrix

$$\tilde{S}_{t+1} = H_{SS}\tilde{S}_t + H_{SZ}\tilde{Z}_{t+1},$$

which is the equilibrium law of motion. If all the eigenvalues are (strictly) within the unit circle, then we will have stable dynamics.

2.3 Example 1: Real-business-cycle (RBC) model

- Preferences:

$$u(C_t) = \frac{C_t^{1-\gamma}}{1-\gamma}.$$

- Production function:

$$Y_t = A_t K_t^{\alpha} L^{1-\alpha},$$

where we normalize $L = 1$.[3]

- Stochastic process for TPF:

$$\ln(A_{t+1}) = \rho \ln(A_t) + \sigma \varepsilon_{t+1},$$

where $\varepsilon \sim \mathrm{N}(0, 1)$.

- Resource constraint (w.l.o.g, for simplicity we assume full depreciation):

$$C_t + K_{t+1} = Y_t.$$

2.3.1 First Step: Find the equations that characterize the equilibrium (optimality, feasibility,...)

The system of equilibrium equations is given by

$$\ln(A_{t+1}) = \rho \ln(A_t) + \sigma \varepsilon_{t+1}, \tag{Eq.0}$$

which is already solved (in the sense that it is already log-linearized) and the equations

$$C_t + K_{t+1} = A_t K_t^{\alpha} \tag{Eq.1}$$

$$C_t^{-\gamma} = \beta \mathbb{E}_t \left[C_{t+1}^{-\gamma} \alpha A_{t+1} K_{t+1}^{\alpha-1} \right] \tag{Eq.2}$$

[3]Note: leisure is not valued.

2.3.2 Second Step: Find the deterministic steady state

Note that (Eq.0) is an AR(1) process, and therefore we can rewrite it as

$$a_t = \rho a_{t-1} + \sigma\varepsilon_t = \rho\left(\rho a_{t-2} + \sigma\varepsilon_{t-1}\right) + \sigma\varepsilon_t = \cdots = \sum_{j=0}^{\infty} \sigma\rho^j \varepsilon_{t-j} = \sigma \sum_{j=0}^{\infty} \rho^j L^j \varepsilon_t.$$

where L is the lag operator. Then, to find the deterministic steady state level of A, we set $\varepsilon_{t-j} = 0$, $\forall j$ obtaining

$$a_t = \sum_{j=0}^{\infty} \sigma\rho^j \varepsilon_{t-j} = 0, \quad \forall t,$$

therefore

$$a_t = \bar{a} = 0 \quad \Longrightarrow \quad \bar{a} = \ln(\bar{A}) \quad \Longrightarrow \quad \bar{A} = e^{\bar{a}} = e^0 = 1.$$

Given $\bar{A} = 1$, then $\bar{Y} = \bar{A}\bar{K}_t^{\alpha} = \bar{K}_t^{\alpha}$ and thus

$$\bar{C} + \bar{K} = \bar{Y} = \bar{K}^{\alpha}, \tag{1ss}$$

$$\bar{C}^{-\gamma} = \beta\mathbb{E}\left[\bar{C}^{-\gamma}\alpha\bar{K}^{\alpha-1}\right] \quad \Longrightarrow \quad \bar{K} = (\alpha\beta)^{\frac{1}{1-\alpha}}. \tag{2ss}$$

Note that substituting (2ss) in (1ss) we obtain

$$\bar{C} = \bar{K}^{\alpha} - \bar{K} = (\alpha\beta)^{\frac{\alpha}{1-\alpha}} - (\alpha\beta)^{\frac{1}{1-\alpha}}.$$

2.3.3 Third Step: Log-linearize the equilibrium equations

For this example, we will use the first method (i.e. we will write the equilibrium equations in log deviations, doing that for each component of S and for each component of X, and finally taking a first-order Taylor approximation of the equilibrium equations in each of the variables $\tilde{s}_t^{(i)}, \tilde{x}_t^{(i)}, \tilde{z}_t^{(i)}$ around $\tilde{s}_t^{(i)} = \tilde{x}_t^{(i)} = \tilde{z}_t^{(i)} = 0$).

From (Eq.0) we can write

$$\ln\left(\bar{A}e^{\tilde{a}_{t+1}}\right) = \rho\ln\left(\bar{A}e^{\tilde{a}_t}\right) + \sigma\varepsilon_{t+1},$$

which implies

$$\ln\left(\bar{A}\right) + \tilde{a}_{t+1} = \rho\left[\ln\left(\bar{A}\right) + \tilde{a}_t\right] + \sigma\varepsilon_{t+1}.$$

As $\bar{A} = 1$, then we have

$$\tilde{a}_{t+1} = \rho\tilde{a}_t + \sigma\varepsilon_{t+1} \tag{0$\ell\ell$}$$

From (Eq.2) we can write

$$\left(\bar{C}e^{\tilde{c}_t}\right)^{-\gamma} = \beta\mathbb{E}_t\left[\left(\bar{C}e^{\tilde{c}_{t+1}}\right)^{-\gamma}\alpha\bar{A}e^{\tilde{a}_{t+1}}\left(\bar{K}e^{\tilde{k}_{t+1}}\right)^{\alpha-1}\right],$$

where rewriting

$$\bar{C}^{-\gamma} e^{-\gamma \tilde{c}_t} = \underbrace{\beta \alpha \bar{A} \bar{C}^{-\gamma} \bar{K}^{\alpha-1}}_{=\bar{C}^{-\gamma} \text{ by } (2ss)} \mathbb{E}_t \left[e^{-\gamma \tilde{c}_{t+1}} e^{\tilde{a}_{t+1}} e^{(\alpha-1)\tilde{k}_{t+1}} \right],$$

and which can finally be expressed as

$$e^{-\gamma \tilde{c}_t} = \mathbb{E}_t \left[e^{-\gamma \tilde{c}_{t+1}} e^{\tilde{a}_{t+1}} e^{(\alpha-1)\tilde{k}_{t+1}} \right] = \mathbb{E}_t \left[e^{-\gamma \tilde{c}_{t+1} + \tilde{a}_{t+1} + (\alpha-1)\tilde{k}_{t+1}} \right]. \tag{4}$$

On the one hand, the Taylor expansion of the LHS of (4) around $\tilde{c}_t = 0$ is given by

$$e^{-\gamma \tilde{c}_t} \cong e^{-\gamma 0} + \left(-\gamma e^{-\gamma \tilde{c}_t} \Big|_{\tilde{c}_t = 0} \right) [\tilde{c}_t - 0] = 1 - \gamma \tilde{c}_t, \tag{5}$$

while, on the other hand, the Taylor expansion of the RHS of (4) around $(\tilde{c}_{t+1}, \tilde{a}_{t+1}, \tilde{k}_{t+1}) = (0, 0, 0)$ is given by

$$\begin{aligned} e^{-\gamma \tilde{c}_{t+1} + \tilde{a}_{t+1} + (\alpha-1)\tilde{k}_{t+1}} &\cong e^0 + \left. \begin{pmatrix} -\gamma e^{-\gamma \tilde{c}_{t+1} + \tilde{a}_{t+1} + (\alpha-1)\tilde{k}_{t+1}} \\ e^{-\gamma \tilde{c}_{t+1} + \tilde{a}_{t+1} + (\alpha-1)\tilde{k}_{t+1}} \\ (\alpha - 1) e^{-\gamma \tilde{c}_{t+1} + \tilde{a}_{t+1} + (\alpha-1)\tilde{k}_{t+1}} \end{pmatrix}' \right|_{(0,0,0)} \begin{bmatrix} \tilde{c}_{t+1} - 0 \\ \tilde{a}_{t+1} - 0 \\ \tilde{k}_{t+1} - 0 \end{bmatrix} \\ &\cong 1 - \gamma \tilde{c}_{t+1} + \tilde{a}_{t+1} + (\alpha - 1)\tilde{k}_{t+1}. \end{aligned} \tag{6}$$

Therefore we can rewrite (4) as

$$1 - \gamma \tilde{c}_t = \mathbb{E}_t \left[1 - \gamma \tilde{c}_{t+1} + \tilde{a}_{t+1} - (1 - \alpha)\tilde{k}_{t+1} \right],$$

where we can substitute ($0\ell\ell$) obtaining

$$1 - \gamma \tilde{c}_t = \mathbb{E}_t \left[1 - \gamma \tilde{c}_{t+1} + \rho \tilde{a}_t + \sigma \varepsilon_t - (1 - \alpha)\tilde{k}_{t+1} \right].$$

Finally, as $\mathbb{E}_t [\varepsilon_t] = 0$, then the only unknown at time t is $\tilde{c}_{t+1}$, and thus we can rewrite the previous equation as

$$-\gamma \tilde{c}_t = \rho \tilde{a}_t - (1 - \alpha)\tilde{k}_{t+1} - \gamma \mathbb{E}_t [\tilde{c}_{t+1}]. \tag{$2\ell\ell$}$$

From (Eq.1) we can write

$$\bar{C} e^{\tilde{c}_t} + \bar{K} e^{\tilde{k}_{t+1}} = \bar{A} e^{\tilde{a}_t} \left(\bar{K} e^{\tilde{k}_t} \right)^{\alpha} = \bar{K}^{\alpha} e^{\tilde{a}_t + \alpha \tilde{k}_t}, \tag{7}$$

as $\bar{A} = 1$. Again we will do a Taylor expansion around the steady state. On the one hand, the Taylor expansion of the LHS of (7) (i.e., $e^{\tilde{c}_t}$ around $\tilde{c}_t = 0$) is almost the same as the one given by (5), where we only have to get rid of the parameter γ. The same applies to the second term of the left-hand-side. On the other hand, the Taylor expansion of the RHS of (7) (i.e., $e^{\tilde{a}_t + \alpha \tilde{k}_t}$) around $(\tilde{a}_t, \tilde{k}_t) = (0, 0)$ is given by

$$e^{\tilde{a}_t + \alpha \tilde{k}_t} \cong e^0 + \left. \begin{pmatrix} e^{\tilde{a}_t + \alpha \tilde{k}_t} \\ \alpha e^{\tilde{a}_t + \alpha \tilde{k}_t} \end{pmatrix}' \right|_{(0,0)} \begin{bmatrix} \tilde{a}_t - 0 \\ \tilde{k}_t - 0 \end{bmatrix} = 1 + \tilde{a}_t + \alpha \tilde{k}_t, \tag{8}$$

and therefore we can write (7) as

$$\bar{C}\left(1+\tilde{c}_t\right)+\bar{K}\left(1+\tilde{k}_{t+1}\right)=\bar{K}^{\alpha}\left(1+\tilde{a}_t+\alpha\tilde{k}_t\right),$$

or, equivalently

$$\underbrace{\bar{C}+\bar{K}-\bar{K}^{\alpha}}_{=0 \text{ by (1ss)}}+\bar{C}\tilde{c}_t+\bar{K}\tilde{k}_{t+1}=+\bar{K}^{\alpha}\tilde{a}_t+\bar{K}^{\alpha}\alpha\tilde{k}_t,$$

where dividing both sides by $\bar{K}^{\alpha}$ yields

$$\bar{C}\bar{K}^{-\alpha}\tilde{c}_t+\bar{K}^{1-\alpha}\tilde{k}_{t+1}=\tilde{a}_t+\alpha\tilde{k}_t. \tag{1$\ell\ell$}$$

To sum up, the log-linearized equations are

$$\tilde{a}_{t+1}=\rho\tilde{a}_t+\sigma\varepsilon_{t+1}, \tag{0$\ell\ell$}$$

$$\bar{C}\bar{K}^{-\alpha}\tilde{c}_t+\bar{K}^{1-\alpha}\tilde{k}_{t+1}=\tilde{a}_t+\alpha\tilde{k}_t, \tag{1$\ell\ell$}$$

$$-\gamma\tilde{c}_t=\rho\tilde{a}_t-(1-\alpha)\tilde{k}_{t+1}-\gamma\mathbb{E}_t\left[\tilde{c}_{t+1}\right]. \tag{2$\ell\ell$}$$

In the spirit of the general model given by (A), we can write (1$\ell\ell$) and (2$\ell\ell$) in matrix form as

$$\begin{bmatrix}\bar{C}\bar{K}^{-\alpha} & -\alpha\\ -\gamma & 0\end{bmatrix}\begin{bmatrix}\tilde{c}_t\\ \tilde{k}_t\end{bmatrix}=\begin{bmatrix}0 & -\bar{K}^{1-\alpha}\\ -\gamma & -(1-\alpha)\end{bmatrix}\mathbb{E}_t\left\{\begin{bmatrix}\tilde{c}_{t+1}\\ \tilde{k}_{t+1}\end{bmatrix}\right\}+\begin{bmatrix}1\\ \rho\end{bmatrix}\tilde{a}_t. \tag{9}$$

2.3.4 Fourth Step: Solve for policy

We use the method of undetermined coefficients, i.e. we make the following guess:

$$\tilde{c}_t=\eta_{ck}\tilde{k}_t+\eta_{ca}\tilde{a}_t, \tag{p1}$$

$$\tilde{k}_{t+1}=\eta_{kk}\tilde{k}_t+\eta_{ka}\tilde{a}_t, \tag{p2}$$

which implies imposing that our controls are linear functions of the state variables of the problem. In the notation of the general model, we guess

$$\begin{bmatrix}\tilde{c}_t\\ \tilde{k}_{t+1}\end{bmatrix}=\begin{bmatrix}\eta_{ck} & \eta_{ca}\\ \eta_{kk} & \eta_{ka}\end{bmatrix}\begin{bmatrix}\tilde{k}_t\\ \tilde{a}_t\end{bmatrix}.$$

From now on we will look for the parameters $\eta_{ck},\eta_{ca},\eta_{kk},\eta_{ka}$.

First we will start with (1$\ell\ell$). Substitute (p1) and (p2) in (1$\ell\ell$) obtaining

$$\bar{C}\bar{K}^{-\alpha}\left(\eta_{ck}\tilde{k}_t+\eta_{ca}\tilde{a}_t\right)+\bar{K}^{1-\alpha}\left(\eta_{kk}\tilde{k}_t+\eta_{ka}\tilde{a}_t\right)=\tilde{a}_t+\alpha\tilde{k}_t,$$

collecting terms in $\tilde{k}_t$ and $\tilde{a}_t$ yields

$$\tilde{k}_t\left(\bar{C}\bar{K}^{-\alpha}\eta_{ck}+\bar{K}^{1-\alpha}\eta_{kk}-\alpha\right)+\tilde{a}_t\left(\bar{C}\bar{K}^{-\alpha}\eta_{ca}+\bar{K}^{1-\alpha}\eta_{ka}-1\right)=0.$$

This equation must hold for all $(\tilde{k}_t, \tilde{a}_t)$. The only way in which this can happen is when both brackets are equal to zero[4]. Then we must have

$$\bar{C}\bar{K}^{-\alpha}\eta_{ck} + \bar{K}^{1-\alpha}\eta_{kk} - \alpha = 0, \tag{H1}$$
$$\bar{C}\bar{K}^{-\alpha}\eta_{ca} + \bar{K}^{1-\alpha}\eta_{ka} - 1 = 0. \tag{H2}$$

Solving for η_{ck} and η_{ca} yields

$$\eta_{ck} = \frac{\alpha - \bar{K}^{1-\alpha}\eta_{kk}}{\bar{C}\bar{K}^{-\alpha}}, \tag{η_{ck}}$$
$$\eta_{ca} = \frac{1 - \bar{K}^{1-\alpha}\eta_{ka}}{\bar{C}\bar{K}^{-\alpha}}. \tag{η_{ca}}$$

Second, we continue with (2ℓℓ). First, we start with the expectation term, where using (p1) we obtain

$$\mathbb{E}_t\left[\tilde{c}_{t+1}\right] = \mathbb{E}_t\left[\eta_{ck}\tilde{k}_{t+1} + \eta_{ca}\tilde{a}_{t+1}\right],$$

and substituting (p2) and (0ℓℓ) yields

$$\begin{aligned}
\mathbb{E}_t\left[\tilde{c}_{t+1}\right] &= \mathbb{E}_t\left[\eta_{ck}\left(\eta_{kk}\tilde{k}_t + \eta_{ka}\tilde{a}_t\right) + \eta_{ca}\left(\rho\tilde{a}_t + \sigma\varepsilon_t\right)\right] \\
&= \mathbb{E}_t\left[\eta_{ck}\left(\eta_{kk}\tilde{k}_t + \eta_{ka}\tilde{a}_t\right)\right] + \mathbb{E}_t\left[\eta_{ca}\left(\rho\tilde{a}_t + \sigma\varepsilon_t\right)\right] \\
&= \eta_{ck}\left(\eta_{kk}\tilde{k}_t + \eta_{ka}\tilde{a}_t\right) + \eta_{ca}\rho\tilde{a}_t \\
&= \eta_{ck}\eta_{kk}\tilde{k}_t + \tilde{a}_t\left(\eta_{ck}\eta_{ka} + \rho\eta_{ca}\right),
\end{aligned} \tag{10}$$

where the second equality follows from the properties of the expectation operator, the third equality from the fact that in the first expectation, everything is known at time t and the same happens in the second expectation, where we also use the fact that ε_t is zero mean. Note that at this stage we obtain multiplicative terms in the η's, which will lead to non-linearities later on. Substituting now (10) in (2ℓℓ) gives us the following expression

$$-\gamma\tilde{c}_t = \rho\tilde{a}_t - (1-\alpha)\tilde{k}_{t+1} - \gamma\left[\eta_{ck}\eta_{kk}\tilde{k}_t + \tilde{a}_t\left(\eta_{ck}\eta_{ka} + \rho\eta_{ca}\right)\right],$$

where we substitute again (p1) and (p2) obtaining

$$-\gamma\left(\eta_{ck}\tilde{k}_t + \eta_{ca}\tilde{a}_t\right) = \rho\tilde{a}_t - (1-\alpha)\left[\eta_{kk}\tilde{k}_t + \eta_{ka}\tilde{a}_t\right] - \gamma\left[\eta_{ck}\eta_{kk}\tilde{k}_t + \tilde{a}_t\left(\eta_{ck}\eta_{ka} + \rho\eta_{ca}\right)\right],$$

collecting the equal terms yields

$$\tilde{k}_t\left[-\gamma\eta_{ck} + (1-\alpha)\eta_{kk} + \gamma\eta_{ck}\eta_{kk}\right] + \tilde{a}_t\left[-\gamma\eta_{ca} - \rho + (1-\alpha)\eta_{ka} + \gamma\left(\eta_{ck}\eta_{ka} + \rho\eta_{ca}\right)\right] = 0.$$

[4]Example: take $\tilde{k}_t = 0$ and $\tilde{a}_t \neq 0$. Then, if the second bracket is different from zero, the condition would not be satisfied.

Again, this equation must hold for all $\tilde{k}_t, \tilde{a}_t$. The only way for this to work is that both brackets are equal to zero. Then we get two more equations since we must have

$$-\gamma\eta_{ck} + (1-\alpha)\eta_{kk} + \gamma\eta_{ck}\eta_{kk} = 0, \tag{H3}$$

$$-\gamma\eta_{ca} - \rho + (1-\alpha)\eta_{ka} + \gamma\eta_{ck}\eta_{ka} + \gamma\rho\eta_{ca} = 0, \tag{H4}$$

The most important parameter is η_{kk}, thus we will solve for it. To this end, take (H3) and divide by γ to obtain

$$-\eta_{ck} + \frac{1-\alpha}{\gamma}\eta_{kk} + \eta_{ck}\eta_{kk} = 0,$$

and now substitute (η_{ck}) to obtain

$$-\frac{\alpha - \bar{K}^{1-\alpha}\eta_{kk}}{\bar{C}\bar{K}^{-\alpha}} + \frac{1-\alpha}{\gamma}\eta_{kk} + \frac{\alpha - \bar{K}^{1-\alpha}\eta_{kk}}{\bar{C}\bar{K}^{-\alpha}}\eta_{kk} = 0.$$

After rearranging some terms we arrive to

$$\alpha\bar{C}^{-1}\bar{K}^{\alpha}\eta_{kk} - \bar{C}^{-1}\bar{K}\eta_{kk}^2 - \alpha\bar{C}^{-1}\bar{K}^{\alpha} + \bar{C}^{-1}\bar{K}\eta_{kk} + \frac{1-\alpha}{\gamma}\eta_{kk} = 0,$$

where multiplying both sides by $\bar{C}$ and further rearranging yields

$$-\bar{K}\eta_{kk}^2 + \eta_{kk}\left(\alpha\bar{K}^{\alpha} + \bar{K} + \frac{(1-\alpha)\bar{C}}{\gamma}\right) - \alpha\bar{K}^{\alpha} = 0.$$

This is a quadratic form that can be solved as

$$\begin{aligned}
\eta_{kk} &= \frac{-\left(\alpha\bar{K}^{\alpha} + \bar{K} + \frac{(1-\alpha)\bar{C}}{\gamma}\right) \pm \sqrt{\left(\alpha\bar{K}^{\alpha} + \bar{K} + \frac{(1-\alpha)\bar{C}}{\gamma}\right)^2 - 4\left(-\bar{K}\right)\left(-\alpha\bar{K}^{\alpha}\right)}}{2\left(-\bar{K}\right)} \\
&= \frac{\alpha\bar{K}^{\alpha} + \bar{K} + \frac{(1-\alpha)\bar{C}}{\gamma} \mp \sqrt{\left(\alpha\bar{K}^{\alpha} + \bar{K} + \frac{(1-\alpha)\bar{C}}{\gamma}\right)^2 - 4\alpha\bar{K}^{1+\alpha}}}{2\bar{K}}.
\end{aligned} \tag{11}$$

In general, this quadratic form will have two real valued solutions (as long as we don't make any crazy calibration). Let's call them $\eta_{kk,1}$ and $\eta_{kk,2}$. Without loss of generality we define $\eta_{kk,1} < \eta_{kk,2}$, where $\eta_{kk,1} \in (0,1)$ and $\eta_{kk,2} > 1$ (this can be shown, not done here). Does this mean that we will have two different solutions that take us to the steady state of the model? Generally the answer is no. Both solutions will fulfil all the equations but one of them, ($\eta_{kk,2}$), will violate the transversality condition. To see this rewrite (p2) as

$$\begin{aligned}
\tilde{k}_t &= \eta_{kk}\tilde{k}_{t-1} + \eta_{ka}\tilde{a}_{t-1} \\
&= \eta_{kk}\left(\eta_{kk}\tilde{k}_{t-2} + \eta_{ka}\tilde{a}_{t-2}\right) + \eta_{ka}\tilde{a}_{t-1} = \eta_{kk}^2\tilde{k}_{t-2} + \eta_{kk}\eta_{ka}\tilde{a}_{t-2} + \eta_{ka}\tilde{a}_{t-1} \\
&= \eta_{kk}^2\left(\eta_{kk}\tilde{k}_{t-3} + \eta_{ka}\tilde{a}_{t-3}\right) + \eta_{kk}\eta_{ka}\tilde{a}_{t-2} + \eta_{ka}\tilde{a}_{t-1} \\
&= \ldots \\
&= \eta_{kk}^t\tilde{k}_0 + \sum_{j=1}^{t}\eta_{kk}^{j-1}\eta_{ka}\tilde{a}_{t-j}
\end{aligned}$$

Therefore, if $\eta_{kk} > 1$ the dynamics will be unstable ($\tilde{k}_t$ would explode) and we will eventually violate the transversality condition given by

$$\lim_{T\to\infty} \mathbb{E}_0 \left[\beta^T K_{T+1} U_C\right] = \lim_{T\to\infty} \mathbb{E}_0 \left[\beta^T K_{T+1} C_T^{-\gamma}\right].$$

2.4 Example 2: Leisure-labour decision

Consider the preferences between consumption C_t and leisure L_t given by

$$u(C_t, L_t) = \frac{C_t^{1-\gamma}}{1-\gamma} + \eta \frac{L_t^{\xi+1}}{\xi+1}.$$

The optimality condition that characterizes the leisure-labour decision is then given by

$$\frac{u_L}{u_C} = \eta \frac{L_t^{\xi}}{C_t^{-\gamma}} = W_t, \tag{12}$$

where W_t is the real wage. For this example, we will use the second method of log-linearization (i.e. take logs of the equilibrium conditions and expand in $\tilde{s}_t^{(i)}, \tilde{x}_t^{(i)}, \tilde{z}_t^{(i)}$). First we compute the deterministic steady state of this problem, which is given by

$$\eta \bar{L}^{\xi} \bar{C}^{\gamma} = \bar{W},$$

which in logs can be expressed as

$$\ln \eta + \xi \ln \bar{L} + \gamma \ln \bar{C} = \ln \bar{W}. \tag{13}$$

Second, taking logs in (12) yields

$$\ln \eta + \xi \ln L_t + \gamma \ln C_t = \ln W_t. \tag{14}$$

Now, subtracting (14) from (13) yields

$$\xi(\ln L_t - \ln \bar{L}) + \gamma(\ln C_t - \ln \bar{C}) = \ln W_t - \ln \bar{W},$$

where, following our usual notation we can write $\tilde{l}_t = \ln(L_t) - \ln(\bar{L})$ (the same applies to C_t and W_t) obtaining

$$\xi \tilde{l}_t + \gamma \tilde{c}_t = \tilde{w}_t.$$

References

Blanchard, O. J. and Kahn, C. M. (1980), 'The solution of linear difference models under rational expectations', *Econometrica* **48**(5).

A Blanchard-Kahn Conditions

This section is based on (Blanchard and Kahn, 1980). Consider the system of equations

$$\begin{bmatrix} X_{t+1} \\ \mathbb{E}_t\left[P_{t+1}\right] \end{bmatrix} = A \begin{bmatrix} X_t \\ P_t \end{bmatrix} + BZ_t, \tag{15}$$

where A is an $(n+m) \times (n+m)$ matrix, B is an $(n+m) \times k$ matrix and

- $X_t \in \mathbb{R}^n$ is a vector of predetermined variables at t (e.g. K_t in the RBC model),
- $P_t \in \mathbb{R}^m$ is a vector of non-predetermined variables at t (e.g. Π_t, Y_t, $\ldots$),
- $Z_t \in \mathbb{R}^k$ is a vector of exogenous shocks.

The difference between predetermined and non-predetermined variables is extremely important. Let Ω_t be the information set at t, which includes past and current values of X, P, Z. A predetermined variable is a function only of variables known at time t, that is of variables in Ω_t such that $X_{t+1} = \mathbb{E}_t\left[X_{t+1}|\Omega_{t+1}\right]$ whatever the realization of any variable in Ω_{t+1}. A non-predetermined variable P_{t+1} can be a function of any variable in Ω_{t+1}, so that we can conclude that $P_{t+1} = \mathbb{E}_t\left[P_{t+1}|\Omega_{t+1}\right]$ only if the realization of all variables in Ω_{t+1} are equal to their expectations conditional on Ω_t.

We further assume that shocks don't explode too fast, i.e. $\forall t$, $\exists \bar{Z}_t \in \mathbb{R}^k$ and $\exists \theta_t \in \mathbb{R}$ such that

$$-(1+i)^{\theta_t}\bar{Z}_t \leqslant \mathbb{E}_t\left[Z_{t+i}|\Omega_t\right] \leqslant (1+i)^{\theta_t}\bar{Z}_t \quad \forall i = 0, 1, 2, \ldots$$

This condition rules out exponential growth of the expectation of Z_{t+i}, held at time t.

Definition A.1 (Solution). *A solution $\{X_t, P_t\}_{t=1}^{\infty}$ is a stochastic sequence of variables in Ω_t which satisfies (15) for any realization of the shocks, $\forall t$ and where expectations don't explode, i.e.*

$$\forall t, \quad \exists \begin{bmatrix} \bar{X}_t \\ \bar{P}_t \end{bmatrix} \in \mathbb{R}^{n+m}, \ \textit{and } \sigma_t \in \mathbb{R},$$

such that

$$-(1+i)^{\sigma_t}\begin{bmatrix} \bar{X}_t \\ \bar{P}_t \end{bmatrix} \leqslant \mathbb{E}_t\left[\begin{matrix} X_{t+i} \\ P_{t+i} \end{matrix}\middle| \Omega_t\right] \leqslant (1+i)^{\sigma_t}\begin{bmatrix} \bar{X}_t \\ \bar{P}_t \end{bmatrix}, \quad \forall i = 0, 1, 2, \ldots \tag{16}$$

Proposition A.1. *Let $\bar{m}$ be the number of eigenvalues of A which lie outside the unit circle (i.e. $|\lambda_i| > 1$). Then*

- *(Blanchard and Kahn, 1980, Proposition 1) If $\bar{m} = m$, i.e. if the number of eigenvalues of A outside the unit circle is equal to the number of non-predetermined variables, then there exists a unique solution.*

- *(Blanchard and Kahn, 1980, Proposition 2) If $\bar{m} > m$, i.e. if the number of eigenvalues outside the unit circle exceeds the number of non-predetermined variables, there is no solution satisfying both (15) and the non-explosion condition.*

- *(Blanchard and Kahn, 1980, Proposition 3) If $\bar{m} < m$, i.e. if the number of eigenvalues outside the unit circle is less than the number of non-predetermined variables, there is an infinity of solutions (i.e., the solution is indeterminate).*

Example A.1. *Suppose the following equilibrium equation for inflation:*

$$\pi_t = \rho \mathbb{E}_t [\pi_{t+1}] + \varepsilon_t, \tag{17}$$

where we assume that ε_t is an i.i.d. error with zero mean. Furthermore, assume that $\rho \geqslant 0$. Let us rewrite it as

$$\mathbb{E}_t [\pi_{t+1}] = \frac{1}{\rho}\pi_t - \frac{1}{\rho}\varepsilon_t.$$

Guess that the solution is of the form

$$\pi_t = C_0 \rho^{-t} + \varepsilon_t, \quad \forall C_0 \in \mathbb{R}, \tag{18}$$

where $C_0 \in \mathbb{R}$ is an arbitrary number. Substituting (18) in (17) we obtain

$$\begin{aligned}
\pi_t &= \rho \mathbb{E}_t \left[C_0 \rho^{-(t+1)} + \varepsilon_{t+1}\right] + \varepsilon_t \\
&= \rho \mathbb{E}_t \left[C_0 \rho^{-(t+1)}\right] + \mathbb{E}_t [\varepsilon_{t+1}] + \varepsilon_t \\
&= \rho C_0 \rho^{-(t+1)} + \varepsilon_t \\
&= C_0 \rho^{-t} + \varepsilon_t.
\end{aligned}$$

Therefore, the solution given by (18) is valid $\forall \rho \geqslant 0$ and $\forall C_0 \in \mathbb{R}$. To evaluate the solution of this model for different values of these parameters, we apply Proposition A.1. In this example, π_t is a non-predetermined variable, thus $m = 1 = \bar{m}$. Furthermore, the matrix A and its only eigenvalue λ is given by the scalar

$$A = \left[\frac{1}{\rho}\right] \implies \lambda = \frac{1}{\rho}.$$

Therefore we have that

- *If $|\rho| < 1$, then $\lambda > 1$ is strictly outside the unit circle and then (17) has a unique solution. In particular, the unique solution that satisfies the non explosive condition (16) is $C_0 = 0$, and therefore $\pi_t = \varepsilon_t$.*

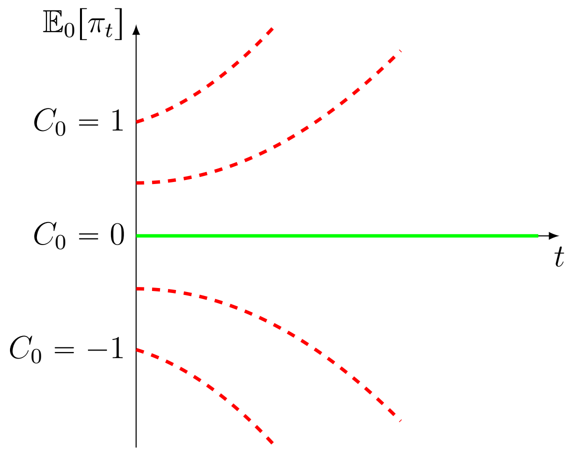

Figure 1: $\rho \in (0, 1)$. There is a unique stable solution, but ∞ unstable solutions. Note that BK rules out hyper-inflationary/deflationary equilibria here! But these may be important equilibria of our model, so this is a word of caution with applying BK blindly (see also Cochrane's critique of the New-Keynesian model).

- *If $|\rho| < 1$, then $\lambda < 1$ is inside the unit circle, and then (17) has multiple solutions*[5]*. In particular, we have an infinite number of solutions (one for each C_0) that satisfy the non explosive condition (16).*

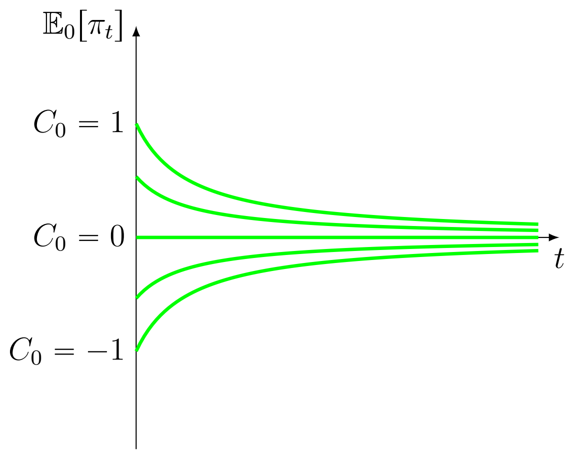

Figure 2: $|\rho| > 1$. There are ∞ stable solutions, i.e., the solution is indeterminate.

Example A.2 (RBC model). *Define*

$$A_0 \equiv \begin{bmatrix} -\alpha & \bar{C}\bar{K}^{-\alpha} \\ 0 & -\gamma \end{bmatrix}, \; A_1 \equiv \begin{bmatrix} -\bar{K}^{1-\alpha} & 0 \\ -(1-\alpha) & -\gamma \end{bmatrix}, \; \textit{and } B_0 \equiv \begin{bmatrix} \rho \\ 1 \end{bmatrix}$$

[5]We call this situation multiplicity of equilibria or indeterminacy.

Then we can rewrite (9) *as*[6]

$$\begin{bmatrix} \tilde{k}_{t+1} \\ \mathbb{E}_t[\tilde{c}_{t+1}] \end{bmatrix} = A \begin{bmatrix} \tilde{k}_t \\ \tilde{c}_t \end{bmatrix} + Ba_t,$$

where

$$A = A_1^{-1}A_0, \text{ and } B = A_1^{-1}B_0.$$

By Proposition A.1, $\bar{m} = 1$, thus this system will have a unique (stable) solution as long as A has exactly one eigenvalue strictly outside the unit circle.

[6]Note that the shock Z_t in (Blanchard and Kahn, 1980) need not be i.i.d.!