Continuous-time dynamic programming

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Abstract

These are notes that I took from the course Macroeconomics II at UC3M, taught by Matthias Kredler during the Spring semester of 2016. Typos and errors are possible, and are my sole responsibility and not that of the instructor.

Contents

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1 (Deterministic) Continuous-Time Dynamic Programming

1.1 Consumption-Savings Problem

Consider the consumption-savings problem in continuous time with a deterministic horizon $t \in [0, T]$ given by

$$\begin{aligned}
&\max_{\{c_t\}_{t=0}^{T}} \int_0^T e^{-\rho t} u(c_t) dt + e^{-\rho T} \bar{V}_T(a_T), \\
&\text{s.t.} \quad \dot{a}_t \left(\equiv \frac{da}{dt} \right) = r a_t + w - c_t, \quad \forall t, \\
&\qquad a_t \geqslant 0, \quad \forall t, \\
&\qquad a_0 \geqslant 0 \text{ given}, \quad a_T \geqslant 0.
\end{aligned} \tag{1}$$

where $\bar{V}_T(\cdot)$ is an exogenously-given final-payoff function such that $\bar{V}_T' > 0$ and $\bar{V}_T'' \leqslant 0$. How is (1) obtained? For a short Δt, we can write

$$\begin{aligned}
a_{t+\Delta t} &= (a_t + w\Delta t - c_t \Delta t)(1 + r\Delta t) \\
&= a_t + r a_t \Delta t + w \Delta t - c_t \Delta t + (wr - rc_t)(\Delta t)^2,
\end{aligned}$$

where $r a_t \Delta t$ is the return earned from savings a_t during the little period Δt; $w\Delta t$ are the wage earnings over Δt; $c_t \Delta t$ are consumption expenditures during Δt and finally $(wr - c_t r)(\Delta t)^2$ are second order terms.[1] Re-arranging the previous equation we obtain

$$a_{t+\Delta t} - a_t = r a_t \Delta t + w\Delta t - c_t \Delta t + (wr - rc_t)(\Delta t)^2,$$

dividing both sides by Δt,

$$\begin{aligned}
a_{t+\Delta t} - a_t &= r a_t \Delta t + w\Delta t - c_t \Delta t + (wr - rc_t)(\Delta t)^2 \\
\frac{a_{t+\Delta t} - a_t}{\Delta t} &= r a_t + w - c_t + (wr - c_t r)\Delta t,
\end{aligned}$$

and finally taking limits as $\Delta t \to 0$ yields

$$\dot{a}_t = r a_t + w - c_t.$$

1.1.1 Bellman's principle

We are going to do a kind of 'backwards induction' to obtain the Hamilton-Jacobi-Bellman equation. To do this, let us assume that we know $V(\bar{t}, a)$, for all $a \geqslant 0$ at some $\bar{t}$. How

[1] Note that a_t is a stock, while w, c_t and $r a_t$ are flows/rates.

can we obtain $V(\bar{t}-\Delta t, a)$? Bellman's principle state that

$$V(\bar{t}-\Delta t, a) = \max_{c\geqslant 0}\left\{\underbrace{\underbrace{u(c)\Delta t}_{\text{`Today'}}}_{[\bar{t}-\Delta t,\bar{t}-\Delta t+\Delta t]} + \underbrace{e^{-\rho\Delta t}V(\bar{t}, a')}_{\text{`Continuation value'}}\right\}, \tag{BP}$$
$$\text{s.t.}\quad a' = a + \left[ra + w - c\right]\Delta t + o\left(\Delta t\right),$$

where $o\left(\Delta t\right)$ are terms such that $\lim_{\Delta t\to 0}\frac{o(\Delta t)}{\Delta t} = 0$ (i.e. of second order and higher). Disregarding $o\left(\Delta t\right)$, we can rewrite (BP) substituting a' as

$$V(\bar{t}-\Delta t, a) = \max_{c\geqslant 0}\left\{u(c)\Delta t + e^{-\rho\Delta t}V(\bar{t}, a + \left[ra + w - c\right]\Delta t)\right\}. \tag{2}$$

Let us define the continuation value as

$$g(\Delta t) \equiv e^{-\rho\Delta t}V\left(\bar{t}, a + \overbrace{\left[ra + w - c\right]}^{=\dot{a}}\Delta t\right). \tag{3}$$

The first-order Taylor expansion of (3) around the point $\Delta t = 0$ is given by

$$g(\Delta t) \cong g(0) + g'(0)\Delta t. \tag{4}$$

To obtain $g'(\Delta t)$ we take the derivative of (3) with respect to Δt, obtaining[2]

$$g'(\Delta t) = -\rho e^{-\rho\Delta t}V\left(\bar{t}, a + \dot{a}\Delta t\right) + e^{-\rho\Delta t}V_a\left(\bar{t}, a + \dot{a}\Delta t\right)\dot{a}.$$

Then we have that $g'(0) = -\rho V\left(\bar{t}, a\right) + \dot{a}V_a\left(\bar{t}, a\right)$. Since $g(0) = V(\bar{t}, a)$, then substituting in (3), the first-order Taylor expansion of (3) around the point $\Delta t = 0$ is given by

$$g(\Delta t) \cong V(\bar{t}, a) + \left(-\rho V\left(\bar{t}, a\right) + \dot{a}V_a\left(\bar{t}, a\right)\right)\Delta t. \tag{5}$$

Going back to (2) we can substitute (5) obtaining

$$V(\bar{t}-\Delta t, a) = \max_{c\geqslant 0}\left\{u(c)\Delta t + V(\bar{t}, a) - \rho V\left(\bar{t}, a\right)\Delta t + \dot{a}V_a\left(\bar{t}, a\right)\Delta t\right\},$$

where substituting again $\dot{a} = ra + w - c$ and rewriting we obtain

$$V(\bar{t}-\Delta t, a) - V(\bar{t}, a) = -\rho V\left(\bar{t}, a\right)\Delta t + \max_{c\geqslant 0}\left\{u(c)\Delta t + \left(ra + w - c\right)V_a\left(\bar{t}, a\right)\Delta t\right\}.$$

Dividing both sides by Δt yields

$$-\frac{V(\bar{t}, a) - V(\bar{t}-\Delta t, a)}{\Delta t} = -\rho V\left(\bar{t}, a\right) + \max_{c}\left\{u(c) + \left(ra + w - c\right)V_a\left(\bar{t}, a\right)\right\},$$

[2] Note that the Value function is defined as $V(t, a)$, thus $V_a(t, a)$ denotes the derivative w.r.t the second argument of this function. In the second part of the derivative we are using the chain rule

and finally, taking limits as $\Delta t \to 0$, and substituting $\bar{t}$ for t since the choice of $\bar{t}$ was arbitrary yields

$$-V_t(t,a) + \rho V(t,a) = \max_{c\geqslant 0} \left\{ u(c) + (ra + w - c) V_a(t,a) \right\}, \tag{HJB}$$

which is called the Hamilton-Jacobi-Bellman (HJB) equation. The unknown in this partial differential equation is $V(t,a)$, where $V : [0,T] \times [0,\infty) \to \mathbb{R}$, with boundary (or final) condition $V(T,a) = \bar{V}_T(a)$ given.

1.1.2 Euler equation

As usual, we are interested in obtaining the Euler equation. Taking the first derivative inside the max operator of (HJB) w.r.t c yields $u_c(c) - V_a(t,a) = 0$, where we assume that the solution will be interior, and thus the optimal policy $c^*(t,a)$ must solve

$$u_c\left(c^*(t,a)\right) = V_a(t,a). \tag{FOC}$$

This tells us that to be optimizing, marginal cost of saving must be equal to marginal benefit. To obtain the Euler equation, we have to differentiate (HJB) w.r.t a (the state variable), and we also have to use (FOC). Note that when considering the Euler equation we always think about optimality, therefore rewriting (HJB) in the optimum we have

$$-V_t(t,a) + \rho V(t,a) = u\left(c^*(t,a)\right) + \left(ra + w - c^*(t,a)\right) V_a(t,a).$$

Assumption. *All the elements on* (HJB) *are differentiable.*

Under Assumption 1.1.2, the derivative of the previous expression w.r.t. a is

$$\begin{aligned} -V_{ta}(t,a) + \rho V_a(t,a) =& u_c\left(c^*(t,a)\right)\frac{dc^*(t,a)}{da} - \frac{dc^*(t,a)}{da}V_a(t,a) - \cdots \\ & \cdots - c^*(t,a)V_{aa}(t,a) + wV_{aa}(t,a) + rV_a(t,a) + arV_{aa}(t,a), \end{aligned}$$

where rewriting yields

$$\begin{aligned} -V_{ta}(t,a) + \rho V_a(t,a) =& \frac{dc^*(t,a)}{da}\left[\underbrace{u_c\left(c^*(t,a)\right) - V_a(t,a)}_{=0 \text{ by (FOC)}}\right] + \cdots \\ & \cdots + \left[\underbrace{ra + w - c^*(t,a)}_{=\dot{a}}\right] V_{aa}(t,a) + rV_a(t,a), \end{aligned}$$

thus the previous expression simplifies to

$$-V_{ta}(t,a) + \rho V_a(t,a) = \dot{a}V_{aa}(t,a) + rV_a(t,a),$$

and rewriting we obtain

$$V_{ta}(t,a) + \dot{a} V_{aa}(t,a) = (\rho - r) V_a(t,a). \tag{6}$$

Now, define an optimal path $a^*(t)$ for the state variable a as

- $a^*(0) = a_0$,
- $\dot{a}^*(t) = r\dot{a}^*(t) + w - c^*(t, a^*(t))$, $\forall t$.

Taking the total derivative of the value function along the optimal path [3] we obtain

$$\frac{\partial}{\partial t}\left[V_a(t, a^*(t))\right] = V_{at}(t, a^*(t)) + V_{aa}(t, a^*(t))\dot{a}^*(t). \tag{7}$$

Now, combining (6) with (7) we obtain

$$\frac{V_{at}(t, a^*(t)) + V_{aa}(t, a^*(t))\dot{a}^*(t)}{V_a(t, a^*(t))} = \frac{\frac{\partial}{\partial t}\left[V_a(t, a^*(t))\right]}{V_a(t, a^*(t))} = \underbrace{\rho - r = \frac{\frac{\partial}{\partial t}\left[u_c(c^*(t, a^*(t)))\right]}{u_c(c^*(t, a^*(t)))}}_{\text{Euler equation}}, \tag{EE}$$

where the last equality comes from (FOC). This is the Euler equation, which tells is that marginal utility grows at rate $\rho - r$.

[3] Intuition: going along the optimal path of a value function in the space (t, a) should always give the left-hand-side of the Euler equation

2 (Stochastic) Continuous-Time Dynamic Programming

2.1 Environment

Consider that $w_t \in \{\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_N\}$ follows a Poisson process with

- Transition rates (Transition hazards): η_{ij}; $i, j \in \{1, ..., N\}$. Interpretation:

$$\Pr(\text{jump to } w_j \text{ within } [t + \Delta t] \,|w_t = w_i) = \eta_{ij}\Delta t + o(\Delta t).$$

- Note: η_{ij} might be > 1!
- Interpretation: jump, on average, every $\dfrac{1}{\eta_{ij}}$ time units.
- $\Pr(2 \text{ jumps: } i \longrightarrow j \longrightarrow k \text{ in } [t; t + \Delta t]) < \eta_{ij}\Delta t \eta_{jk}\Delta t = o(\Delta t)$

2.2 Consumption-Savings Problem

Suppose $w_t \in \{\bar{w}_1, \bar{w}_2\}$ and $\eta_{12} = \eta_{21} = \eta$.

2.2.1 Bellman's principle

Suppose first that the wage at some $\bar{t} - \Delta t$ is known and is $\bar{w}_1$, then we can write

$$V(\bar{t} - \Delta t, a, \bar{w}_1) = \max_{c \geqslant 0} \left\{ u(c)\Delta t + e^{-\rho \Delta t} \overbrace{\mathbb{E}_{\bar{t}-\Delta t}\left[V(\bar{t}, a', w')\right]}^{=\mathbb{E}\left[V(\bar{t},a',w')|w_{\bar{t}-\Delta t}=\bar{w}_1\right]} \right\},$$

$$\text{s.t.} \quad a' = a + \underbrace{[ra + \bar{w}_1 - c]}_{\dot{a}} \Delta t + \underbrace{o(\Delta t)}_{\substack{2^{nd} \\ \text{order terms}}}, \tag{BC}$$

$$w' = w_{\bar{t}} \quad \text{stochastic (exogenous).}$$

Working on the expectation we can write the previous equation as

$$V(\bar{t} - \Delta t, a, \bar{w}_1) = \max_{c \geqslant 0} \left\{ u(c)\Delta t + e^{-\rho \Delta t} \left[\begin{array}{c} \underbrace{(1 - \eta \Delta t)V(\bar{t}, a', \bar{w}_1)}_{\text{Probability } w \text{ doesn't change}} + \cdots \\ \cdots + \underbrace{\eta \Delta t V(\bar{t}, a', \bar{w}_2)}_{\text{Probability } w \text{ changes}} + o(\Delta t) \end{array} \right] \right\} \tag{BP}$$

Disregarding $o(\Delta t)$, we can define

$$g(\Delta t) \equiv e^{-\rho \Delta t} \left[(1 - \eta \Delta t)V(\bar{t}, a', \bar{w}_1) + \eta \Delta t V(\bar{t}, a', \bar{w}_2)\right],$$

where substituting a' from (BC) yields

$$g(\Delta t) = e^{-\rho\Delta t}\left[(1-\eta\Delta t)V(\bar{t}, a+\dot{a}\Delta t, \bar{w}_1) + \eta\Delta t V(\bar{t}, a+\dot{a}\Delta t, \bar{w}_2)\right]. \tag{8}$$

As in the deterministic case, the Taylor expansion of $g(\Delta t)$ around $\Delta t = 0$ is given by

$$g(\Delta t) \cong g(0) + g'(0)\Delta t, \tag{9}$$

where, on the one hand, the term $g(0)$ is obtained by setting $\Delta t = 0$ in (8), which gives

$$g(0) = V(\bar{t}, a, \bar{w}_1). \tag{10}$$

On the other hand we have

$$\begin{aligned} g'(\Delta t) = &-\rho\, \overbrace{e^{-\rho\Delta t}\Big[(1-\eta\Delta t)V(\bar{t}, a+\dot{a}\Delta t, \bar{w}_1) + \eta\Delta t V(\bar{t}, a+\dot{a}\Delta t, \bar{w}_2)\Big]}^{=g(\Delta t)} + \cdots \\ &\cdots + e^{-\rho\Delta t}\Big[-\eta V(\bar{t}, a+\dot{a}\Delta t, \bar{w}_1) + (1-\eta\Delta t)V_a(\bar{t}, a+\dot{a}\Delta t, \bar{w}_1)\dot{a} + \cdots \\ &\cdots + \eta V(\bar{t}, a+\dot{a}\Delta t, \bar{w}_2) + \eta\Delta t V_a(\bar{t}, a+\dot{a}\Delta t, \bar{w}_2)\dot{a}\Big]. \end{aligned} \tag{11}$$

Evaluating this expression at $\Delta t = 0$ yields

$$\begin{aligned} g'(0) &= -\rho e^{-0} g(0) + e^{-0}\left[-\eta V(\bar{t}, a, \bar{w}_1) - (1-0)V_a(\bar{t}, a, \bar{w}_1)\dot{a} + \eta V(\bar{t}, a, \bar{w}_2) + 0\right] \\ &= -\rho V(\bar{t}, a, \bar{w}_1) + V_a(\bar{t}, a, \bar{w}_1)\dot{a} + \eta\left[V(\bar{t}, a, \bar{w}_2) - V(\bar{t}, a, \bar{w}_1)\right]. \end{aligned} \tag{12}$$

Here we can see that the change in the assets when salary changes from $\bar{w}_1$ to $\bar{w}_2$ is of second order, and thus we obtain a 0 when evaluating at Δt.[4] Finally, combining (10) and (12) yields[5]

$$\begin{aligned} g(\Delta t) &\cong V(\bar{t}, a, \bar{w}_1) + \left\{-\rho V(\bar{t}, a, \bar{w}_1) + V_a(\bar{t}, a, \bar{w}_1)\dot{a} + \eta\left[V(\bar{t}, a, \bar{w}_2) - V(\bar{t}, a, \bar{w}_1)\right]\right\}\Delta t \\ &\cong V(\bar{t}, a, \bar{w}_1) - \underbrace{\rho V(\bar{t}, a, \bar{w}_1)\Delta t}_{\text{Discounting}} + \underbrace{\eta\left[V(\bar{t}, a, \bar{w}_2) - V(\bar{t}, a, \bar{w}_1)\right]\Delta t}_{\text{Wage risk}} + \cdots \\ &\quad \cdots + \underbrace{\dot{a}V_a(\bar{t}, a, \bar{w}_1)\Delta t}_{\text{Benefit from saving}}. \end{aligned} \tag{13}$$

[4]From (11), this change is given by the expression coming from

$$-\eta\Delta t V_a(\bar{t}, a+\dot{a}\Delta t, \bar{w}_1)\dot{a} + \eta\Delta t V_a(\bar{t}, a+\dot{a}\Delta t, \bar{w}_2)\dot{a},$$

which can be re-arranged as

$$\eta\Delta t\dot{a}\left[V_a(\bar{t}, a+\dot{a}\Delta t, \bar{w}_2) - V_a(\bar{t}, a+\dot{a}\Delta t, \bar{w}_1)\right].$$

[5]Note: The effects of saving in the other wage state, $\bar{w}_2$, are of second order, that is, only $V_a(\cdot, \bar{w}_1)$ matters in the limit.

Once we have found the Taylor approximation we can go back to (BP), where substituting (13) yields

$$V(\bar{t}-\Delta t, a, \bar{w}_1) = \max_{c\geqslant 0}\left\{\begin{aligned} & u(c)\Delta t + V(\bar{t}, a, \bar{w}_1) - \rho V(\bar{t}, a, \bar{w}_1)\Delta t + \cdots \\ & \quad \cdots + \eta\left[V(\bar{t}, a, \bar{w}_2) - V(\bar{t}, a, \bar{w}_1)\right]\Delta t + \dot{a}V_a(\bar{t}, a, \bar{w}_1)\Delta t \end{aligned}\right\},$$

where further rewriting and substituting $\dot{a}$ from the budget constraint (BC) yields

$$\begin{aligned} & V(\bar{t}-\Delta t, a, \bar{w}_1) - V(\bar{t}, a, \bar{w}_1) + \rho V(\bar{t}, a, \bar{w}_1)\Delta t = \cdots \\ & \quad \cdots = \eta\left[V(\bar{t}, a, \bar{w}_2) - V(\bar{t}, a, \bar{w}_1)\right]\Delta t + \max_{c\geqslant 0}\left\{u(c)\Delta t + \left[ra + \bar{w}_1 - c\right]V_a(\bar{t}, a, \bar{w}_1)\Delta t\right\}. \end{aligned}$$

Dividing by Δt we obtain

$$\begin{aligned} & \frac{V(\bar{t}-\Delta t, a, \bar{w}_1) - V(\bar{t}, a, \bar{w}_1)}{\Delta t} + \rho V(\bar{t}, a, \bar{w}_1) = \cdots \\ & \quad \cdots = \eta\left[V(\bar{t}, a, \bar{w}_2) - V(\bar{t}, a, \bar{w}_1)\right] + \max_{c\geqslant 0}\left\{u(c) + \left[ra + \bar{w}_1 - c\right]V_a(\bar{t}, a, \bar{w}_1)\right\}, \end{aligned}$$

and taking limits when $\Delta t \to 0$ in the previous expression gives

$$\begin{aligned} & \lim_{\Delta t\to 0}\left\{\frac{V(\bar{t}-\Delta t, a, \bar{w}_1) - V(\bar{t}, a, \bar{w}_1)}{\Delta t} + \rho V(\bar{t}, a, \bar{w}_1)\right\} = \cdots \\ & \quad \cdots = \lim_{\Delta t\to 0}\left\{\eta\left[V(\bar{t}, a, \bar{w}_2) - V(\bar{t}, a, \bar{w}_1)\right] + \max_{c\geqslant 0}\left\{u(c) + \left[ra + \bar{w}_1 - c\right]V_a(\bar{t}, a, \bar{w}_1)\right\}\right\}, \end{aligned}$$

where, by the definition of the derivative,

$$\begin{aligned} \lim_{\Delta t\to 0}\frac{V(\bar{t}-\Delta t, a, \bar{w}_1) - V(\bar{t}, a, \bar{w}_1)}{\Delta t} &= -\frac{V(\bar{t}, a, \bar{w}_1) - V(\bar{t}-\Delta t, a, \bar{w}_1)}{\Delta t} \\ &= -V_t(\bar{t}, a, \bar{w}_1), \end{aligned}$$

and as $\bar{t}$ was arbitrary, rewriting one more time we finally obtain

$$\begin{aligned} -V_t(t, a, \bar{w}_1) + \rho V(t, a, \bar{w}_1) = & \eta\left[V(t, a, \bar{w}_2) - V(t, a, \bar{w}_1)\right] + \cdots \\ & \cdots + \max_{c\geqslant 0}\left\{u(c) + \left[ra + \bar{w}_1 - c\right]V_a(t, a, \bar{w}_1)\right\}, \end{aligned} \tag{HJB1}$$

which is our desired Hamilton-Jacobi-Bellman (HJB) equation.

Note that now we have another symmetric Hamilton-Jacobi-Bellman (HJB) equation[6] for $\bar{w}_2$ (i.e. $V(\cdot, \bar{w}_2)$), so switching $\bar{w}_1$ and $\bar{w}_2$ we get

$$\begin{aligned} -V_t(t, a, \bar{w}_2) + \rho V(t, a, \bar{w}_2) = & \eta\left[V(t, a, \bar{w}_1) - V(t, a, \bar{w}_2)\right] + \cdots \\ & \cdots + \max_{c\geqslant 0}\left\{u(c) + \left[ra + \bar{w}_2 - c\right]V_a(t, a, \bar{w}_2)\right\}. \end{aligned} \tag{HJB2}$$

Both equations (HJB1) and (HJB2) form a system of two partial differential equations that allow us to solve for $V(t, a, \bar{w}_1)$ and $V(t, a, \bar{w}_2)$, given terminal conditions $V(T, a, \bar{w}_1)$ and $V(T, a, \bar{w}_2)$.

[6] Since $\bar{w}_1$ was arbitrarily chosen, in principle we can have as many (HJB) as different states of nature we face.

2.2.2 Stationary Case

Let $T = \infty$. Then the value function $V(t, a, w)$ becomes time-independent. In this case we get the following Hamilton-Jacobi-Bellman (HJB) Equation

$$\rho V(a, w) = \eta \left[V(a, \tilde{w}) - V(a, w) \right] + \max_{c \geqslant 0} \left\{ u(c) + \left[ra + w - c \right] V_a(a, w) \right\}, \tag{HJBS}$$

where $\tilde{w} \neq w$ denotes the other earnings level. This is the Hamilton-Jacobi-Bellman (HJB) Equation for the stationary case.

What can we say about the solution of this PDE? As usual, the first-order condition of (HJBS) for the optimal consumption rule $c^*(a, w)$ is

$$u'(c^*(a, w)) = V_a(a, w). \tag{FOC}$$

The stochastic process for wealth under the optimal savings rule is

$$\dot{a}_t^* = ra_t^* + w_t - c^*(a_t^*, w_t). \tag{14}$$

We want to obtain the Euler equation, restricting out attention to the case $a_t > 0$ in which the no-borrowing limit ($a_t \geqslant \bar{a} = 0$) is not binding. To do so, we take the derivative of (HJBS) with respect to the state variable a. To this end, we first get rid of the max operator by evaluating (HJBS) at the optimum given by (FOC), obtaining

$$\rho V(a^*, w) = \eta \left[V(a^*, \tilde{w}) - V(a^*, w) \right] + u(c^*(a^*, w)) + \left[ra^* + w - c^*(a^*, w) \right] V_a(a^*, w),$$

so that the derivative is

$$\begin{aligned} \rho V_a(a^*, w) =& \eta \left[V_a(a^*, \tilde{w}) - V_a(a^*, w) \right] + \frac{\partial u(c^*(a^*, w))}{\partial a^*} + \cdots \\ & \cdots + \left[r - \frac{\partial c^*(a^*, w)}{\partial a^*} \right] V_a(a^*, w) + \left[ra^* + w - c^*(a^*, w) \right] V_{aa}(a, w). \end{aligned} \tag{15}$$

Applying the chain rule, we have

$$\frac{\partial u(c^*(a^*, w))}{\partial a^*} = \frac{\mathrm{d}u(c^*(a^*, w)}{\mathrm{d}c^*(a^*, w)} \frac{\partial c^*(a^*, w)}{\partial a^*} = u'(c^*(a^*, w) \frac{\partial c^*(a^*, w)}{\partial a^*},$$

and substituting this expression in the previous equation and re-arranging we obtain

$$\begin{aligned} \rho V_a(a^*, w) =& \eta \left[V_a(a^*, \tilde{w}) - V_a(a^*, w) \right] + \frac{\partial c^*(a^*, w)}{\partial a^*} \overbrace{\left[u'(c^*(a^*, w)) - V_a(a^*, w) \right]}^{=0 \text{ by (FOC)}} + \cdots \\ & \cdots + rV_a(a^*, w) + \left[ra^* + w - c^*(a^*, w) \right] V_{aa}(a^*, w), \end{aligned}$$

and finally we arrive to

$$\rho V_a(a^*, w) = \eta \left[V_a(a^*, \tilde{w}) - V_a(a^*, w) \right] + rV_a(a^*, w) + \left[ra^* + w - c^*(a^*, w) \right] V_{aa}(a^*, w). \tag{16}$$

Note that now we are in a stochastic framework, and so we can't follow the characteristic optimal path like we did in the deterministic case, because now we can't know in which path we are due to the uncertain environment. To be able to solve this, we will first take a look at the Euler equation in a stochastic framework in discrete time. Why is this interesting? In principle, continuous time and discrete time only make a difference in the mathematical background we use, but the underlying economic theory is the same and thus we should arrive at the same conclusions. The Euler equation in a stochastic framework in discrete time is given by

$$u'(c_t) = \beta R \mathbb{E}\left[u'(c_{t+1})\right],$$

or equally

$$\frac{\mathbb{E}\left[u'(c_{t+1})\right]}{u'(c_t)} = \frac{1}{\beta R}.$$

The interpretation is as usual, the expected growth rate of marginal utility is equal to $(\beta R)^{-1}$. The crucial question we have to answer now is what is the growth rate of marginal utility in continuous time. To address this question, we first need to introduce the following definition.

Definition 2.1 (Infinitesimal generator). *We define the infinitesimal generator $\mathcal{A}$ of an arbitrary function $f(\cdot)$ of the state as the following expected (total) time derivative*

$$\mathcal{A}f(a_t^*, w_t) = \lim_{\Delta t \to 0} \frac{\mathbb{E}_t\left[f(a_{t+\Delta t}^*, w_{t+\Delta t})\right] - f(a_t^*, w_t)}{\Delta t}, \tag{17}$$

where assets a_t^* follow the law of motion specified by the optimal savings rule (14) and w_t follows its exogenous law of motion.

In our stochastic framework, the expectation term of (17) can be written as

$$\begin{aligned}\mathbb{E}_t\left[f\left(a_{t+\Delta t}^*, w_{t+\Delta t}\right)\right] =& \mathbb{E}_t\left[f\left(a'^*, w'\right)\right] \\ &\cong (1-\eta\Delta t)\, f\left(a_t^* + \dot{a}^*\Delta t, w_t\right) + \cdots \\ &\quad \cdots + \eta\Delta t f\left(a_t^* + \dot{a}^*\Delta t, \tilde{w}_t\right) + o(\Delta t).\end{aligned} \tag{18}$$

The first-order Taylor approximations of $f(a_t + \dot{a}\Delta t, w_t)$ and $f(a_t + \dot{a}\Delta t, \tilde{w}_t)$ around the point $\Delta t = 0$ are given by

$$\begin{aligned}f(a_t^* + \dot{a}^*\Delta t, \omega) &\cong f(a_t^*, \omega) + \dot{a}^* f_a\left(a_t^* + \dot{a}^*\Delta t, \omega\right)\big|_{\Delta t=0}\left[\Delta t - 0\right] \\ &= f(a_t^*, \omega) + \dot{a}^*\Delta t f_a\left(a_t^*, \omega\right),\end{aligned} \tag{19}$$

where $\omega = \{w_t, \tilde{w}_t\}$. Substituting (19) in (18) yields

$$
\begin{aligned}
\mathbb{E}_t \left[f\left(a^*_{t+\Delta t}, w_{t+\Delta t}\right)\right] &\cong (1-\eta\Delta t)\left[f\left(a^*_t, w_t\right) + \dot{a}^*\Delta t f_a\left(a^*_t, w_t\right)\right] + \cdots \\
&\quad \cdots + \eta\Delta t\left[f\left(a^*_t, \tilde{w}_t\right) + \dot{a}^*\Delta t f_a\left(a^*_t, \tilde{w}_t\right)\right] \\
&\cong f\left(a^*_t, w_t\right) - \eta\Delta t f\left(a^*_t, w_t\right) + \dot{a}^*\Delta t f_a\left(a^*_t, w_t\right) - \cdots \\
&\quad \cdots - \underbrace{\eta\dot{a}^*(\Delta t)^2 f_a\left(a^*_t, w_t\right)}_{o(\Delta t)} + \eta\Delta t f\left(a^*_t, \tilde{w}_t\right) + \underbrace{\eta\dot{a}^*(\Delta t)^2 f_a\left(a^*_t, \tilde{w}_t\right)}_{o(\Delta t)}.
\end{aligned}
$$

Disregarding the second order terms we obtain

$$
\mathbb{E}_t \left[f\left(a^*_{t+\Delta t}, w_{t+\Delta t}\right)\right] \cong f\left(a^*_t, w_t\right) - \eta\Delta t f\left(a^*_t, w_t\right) + \dot{a}^*\Delta t f_a\left(a^*_t, w_t\right) + \eta\Delta t f\left(a^*_t, \tilde{w}_t\right), \tag{20}
$$

and now we can substitute (20) in (17) to obtain

$$
\begin{aligned}
\mathcal{A}f(a,w) &= \lim_{\Delta t\to 0} \frac{f\left(a^*_t, w_t\right) - \eta\Delta t f\left(a^*_t, w_t\right) + \dot{a}^*\Delta t f_a\left(a^*_t, w_t\right) + \eta\Delta t f\left(a^*_t, \tilde{w}_t\right) - f(a^*_t, w_t)}{\Delta t} \\
&= -\eta f\left(a^*_t, w_t\right) + \dot{a}^* f_a\left(a^*_t, w_t\right) + \eta f\left(a^*_t, \tilde{w}_t\right),
\end{aligned}
$$

where re-arranging yields

$$
\mathcal{A}f(a^*_t, w_t) = \underbrace{\dot{a}^*_t f_a\left(a^*_t, w_t\right)}_{\text{Drift in a}} + \underbrace{\eta\left[f\left(a^*_t, \tilde{w}_t\right) - f\left(a^*_t, w_t\right)\right]}_{\text{Wage risk (hazard rate)}}. \tag{21}
$$

Remark. *Formally, $\mathcal{A}$ is an operator that maps functions $f(a,w)$ that are continuously differentiable in a into functions $g(a,w)$ that are continuous in a. In stochastic processes, the infinitesimal generator is an object that is uniquely associated to a stochastic process (here: the process $\{a^*_t, w_t\}_{t=0}^{\infty}$) that tells us about the properties of the process. In mathematical terms, we could write the operator as*

$$
\mathcal{A} = \dot{a}^* \frac{\partial}{\partial a} + \eta\Delta_w,
$$

where we define the operator Δ_w as the 'discrete derivative in dimension w', $\Delta_w f(\cdot, w) \equiv f(\cdot, \tilde{w}) - f(\cdot, w)$. This expresses how the operator $\mathcal{A}$ acts on functions f (and transforms them into functions g) and has a very clear intuition (expected time change!). In settings with shocks of Brownian-Motion type to continuous variables (here: a) would show up as second-derivative terms, $f_{aa}(a,w)$, in the infinitesimal generator.

Back to (16) note that $V_{aa} = (V_a)_a$, and thus re-arranging (16) we obtain

$$
(\rho - r)\, V_a(a^*, w) = \eta\left[V_a(a^*, \tilde{w}) - V_a(a^*, w)\right] + \left[ra^* + w - c\right](V_a)_a(a^*, w).
$$

If we define $f \equiv V_a$ and bearing in mind that $[ra + w - c] = \dot{a}$, then using (21) we can rewrite the previous expression as

$$
(\rho - r)\, V_a(a^*, w) = \mathcal{A}V_a(a^*, w),
$$

and thus we finally obtain

$$\underbrace{\frac{\mathcal{A}V_a(a^*, w)}{V_a(a^*, w)}}_{\substack{\text{Expected \%} \\ \text{growth rate of } V_a}} = \rho - r,$$

or using (FOC) again

$$\frac{\mathcal{A}u'(c^*(a^*, w))}{u'(c^*(a^*, w))} = \rho - r.$$

3 Pontryagin's Maximum Principle (PMP)

$$\begin{aligned}\max_{\{u_t\}_{t=0}^{T}} \quad & \int_0^T L(x_t,u_t,t)dt + M(x_T)\\ \text{s.t.} \quad & \dot{x}_t = f(x_t,u_t,t),\\ & x_0 \text{ given}, \quad x_t \in \mathbb{R}^n.\end{aligned}$$

The link between this generic notation and the deterministic consumption-savings problem is as follows:

$$x_t \hat{=} a_t, \quad u_t \hat{=} c_t, \quad L(x_t,u_t,t) \hat{=} e^{-\rho t}u(c_t), \quad M \hat{=} V_T, \quad f(x_t,u_t,t) \hat{=} ra + w - c, \quad \lambda \hat{=} V_a.$$

We define the **Hamiltonian** as

$$H(x_t,u_t,t,\lambda_t) = \underset{1\times n}{\lambda_t'}\, \underset{n\times 1}{f(x_t,u_t,t)} + L(x_t,u_t,t),$$

where $\lambda_t \in \mathbb{R}^n$ (has the same dimensionality as the state) and we will call it **co-state variable**. Pontryagin's Maximum Principle tells us that that the optimal control u^* with associated trajectories x^* and λ^* satisfies the following equations:

$$\begin{aligned} u_t^* &= \arg\max_u \underbrace{H(x_t^*,u_t,t,\lambda_t^*)}_{\hat{=}\left(\underbrace{\max_c \{u(c) + \dot{a}V_a\}}_{\text{right-hand-side of (HJB)}}\right)}, \quad \forall t,\\ -\dot{\lambda}_t^* &= H_x(x_t^*,u_t^*,t,\lambda_t^*), \quad \forall t, \qquad \text{(co-state equation } \hat{=}\text{(EE))}\\ \lambda_T^* &= M_x(x_T).\end{aligned}$$

To be able to apply Pontryagin's Maximum Principle to the savings problem, we first write the Hamiltonian associated to this problem:

$$H(a,c,t;V_a) = e^{-\rho t}u(c) + V_a\left(ra + w - c\right),$$

where $V_a \hat{=} \lambda_t$, $e^{-\rho t}u(c_t) \hat{=} L(x_t,u_t,t)$ and $(ra + w - c) \hat{=} f(x_t,u_t,t)$. Then, applying Pontryagin's Maximum principle we have that the first derivative w.r.t to the state variable a yields

$$-\dot{\lambda}_t^* = H_a(a,c,t;V_a) = V_a r \iff -\dot{\lambda}_t^* = r\lambda_t^* \quad (\equiv -\dot{V}_a^* = rV_a^*). \tag{22}$$

Accordingly, the first derivative w.r.t to the control variable c yields

$$H_c(a,c,t;V_a) = 0 \iff e^{-\rho t}u'(c) - \lambda_t = 0 \iff e^{-\rho t}u'(c) = \lambda_t \quad (\equiv e^{-\rho t}u_c(c) = V_a). \tag{23}$$

Taking the derivative w.r.t time in the previous equation yields

$$-\rho e^{-\rho t}u_c(c_t)+e^{-\rho t}\frac{d}{dt}\left[u_c(c^*)\right]=\dot{\lambda}_t \iff e^{-\rho t}\left(\frac{d}{dt}\left[u_c(c^*)\right]-\rho u_c(c_t)\right)=\dot{\lambda}_t. \tag{24}$$

Combining (22) - (24) we obtain

$$-e^{-\rho t}\left(\frac{d}{dt}\left[u_c(c^*)\right]-\rho u_c(c_t)\right)=re^{-\rho t}u_c(c),$$

which simplifies to

$$(\rho-r)\,u_c(c)=\frac{d}{dt}\left[u_c(c^*)\right],$$

and can be rewritten as

$$\frac{\frac{d}{dt}\left[u_c(c^*)\right]}{u_c(c^*)}=\rho-r, \tag{25}$$

which gives us the same Euler equation as in (EE).

Example 3.1 (Log utility). *Consider $u(c)=\ln(c)$. To solve for the optimal path of consumption as a function of time, $c^{*,p}(t)$, from (25) we have*

$$\frac{d}{dt}\left(\frac{1}{c^{*,p}(t)}\right)=(\rho-r)\frac{1}{c^{*,p}(t)},$$

that yields

$$-\left(\frac{1}{\left(c^{*,p}(t)\right)^2}\right)\dot{c}^{*,p}(t)=(\rho-r)\frac{1}{c^{*,p}(t)}.$$

Simplifying this expression gives

$$-\dot{c}^{*,p}(t)=(\rho-r)c^{*,p}(t),$$

which characterizes the growth rate of consumption.