Dynamic Programming under Uncertainty

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(based on Matthias Kredler's lectures)

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Abstract

These are notes that I took from the course Macroeconomics II at UC3M, taught by Matthias Kredler during the Spring semester of 2016. Typos and errors are possible, and are my sole responsibility and not that of the instructor.

Contents

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1 General Framework

In this section we provide an extension of the previous setting. Now we are going to assume that the agent does not now exactly the next period values. However, she will have an expectation over it depending on a shock, i.e., she will be able to forecast tomorrow's values.

1.1 Notation

Consider a general stochastic environment with infinite horizon. There is a shock vector $z_t \in \mathbb{R}^{N_z}$, (e.g. $N_z = 1$, thus $z_t \in \{\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_N\}$ where each $\bar{z}_i \in \mathbb{R}$). We typically assume that the transition from one period to the next one is given by

- An i.i.d. process, i .e., $Pr(z_{t+1} \mid z_t, z_{t-1}, \ldots) = Pr(z_{t+1})$.[1]
- A first-order Markov process, i .e., $Pr(z_{t+1} \mid z_t, z_{t-1}, \ldots) = Pr(z_{t+1} \mid z_t)$.[2]

Henceforth, we assume z_t follows the latter process. Besides, suppose we are given a vector $y_t \in \mathbb{R}^{N_y}$ which informs us about the feasible set for a control $u_t \in \mathbb{R}^{N_u}$ and the return in period t. Specifically, we are given a feasibility correspondence Γ:

$$u_t \in \Gamma(y_t, z_t), \quad \Gamma : \mathbb{R}^{N_y+N_z} \rightrightarrows \mathbb{R}^{N_u},$$

and a return function, F:

$$F(y_t, z_t; u_t), \quad F : \mathbb{R}^{N_y+N_z+N_u} \to \mathbb{R}.$$

[1]These processes are particularly useful as they allow us to reduce the dimensionality of the state vector. In particular, we will not need to condition the expectation of the next period value function on the current shock since it does not contain any useful information related to tomorrow's shock.

[2]First-order Markov processes are also useful as they tell us that all we need to know to be able to forecast tomorrow's shock is the shock observed today.

Finally, suppose that we are given a law of motion for y, i.e. a function h that tells us which value y takes tomorrow:

$$y_{t+1} = h(u_t, z_t; u_t, z_{t+1}), \quad h : \mathbb{R}^{N_y+N_z+N_u+N_z} \to \mathbb{R}^{N_y}.$$

Given this environment we can always write down a valid functional equations for a value function $V(\cdot)$ that reads:

$$V(y, z) = \max_{u \in \Gamma(y,z)} \left\{ F(y, z; u) + \beta \int V(z', h(y, z; u, z'))f(z'|z)dz' \right\}. \tag{1}$$

However, we will see that this formulation may be wasteful. It may be that we can condense the *state* of the economy into a vector$x$ of lower dimensionality that (y, z). This turns out to be of huge value for both analytical and computational purposes.

Definition 1.1 (**State**)**.** *The state of the economy is the smallest set of variables, a vector $x \in \mathbb{R}^{N_x}$, $N_x \leqslant N_y + N_z$, that allows us to determine all of the following:*

1. *the feasible set, i.e. $\tilde{\Gamma} = \Gamma(y, z)$ for some correspondence $\tilde{\Gamma} : \mathbb{R} : N_x \rightrightarrows \mathbb{R}^{N_u}$,*

2. *the return function given a control vector u, i.e. $\tilde{F}(x; u) = F(y, z; u)$, for some function $F : \mathbb{R}^{N_x+N_u} \to \mathbb{R}$,*

3. *the law of motion for y given control vector u and shock vector z', i.e. $y' = \tilde{h}(x; u, z')$ for some function $\tilde{h} : \mathbb{R}^{N_x+N_u+N_z} \Rightarrow \mathbb{R}^{N_x}$,*

4. *and the conditional expectation in the Bellman equation, i.e., it has to hold that $\tilde{f}(z'|x) = f(z'|z)$, for some function $\tilde{f}$.*[3]

From (1) the Bellman equation using the state x is then

$$\tilde{V}(x) = \max_{u \in \tilde{\Gamma}(x)} \left\{ \tilde{F}(x; u) + \beta \int \tilde{V}(z', \tilde{h}(x; u, z'))\tilde{f}(z'|x)dz' \right\}.$$

1.2 Examples

1.2.1 Example 1: Stochastic growth with productivity shocks

Consider a standard neoclassical growth model in which output is produced according to the production function $y_t = A_t k_t^\alpha$ where A_t is a productivity shock that follows a first-order Markov process with conditional density $f(A_{t+1}|A_t)$, k_t is capital at t. The

[3]This condition says that knowledge about the state allows us to make the best possible forecast for tomorrow.

investment in standard, the planner has to choose tomorrow's capital (the *control*) given a feasible-set correspondence

$$K_{t+1} \in \Gamma(k_t, A_t) \equiv [0, A_t k_t^\alpha + (1-\delta)k_t].$$

The period-t return to the planner is

$$F(k_t, A_t; k_{t+1}) = \ln\left(A_t k_t^\alpha + (1-\delta)k_t - k_{t+1}\right).$$

The planner discounts the future at factor $\beta \in (0,1)$. Therefore the dynamic programming form of this problem is given by

- State: k, A.
- Control: k',
- Feasible set correspondence: $\Gamma(k, A) = [0, Ak^\alpha + (1-\delta)k]$,
- Return function: $F(k, A; k') = \ln(Ak^\alpha + (1-\delta)k + k')$,
- Law of motion: $(k', A') = h(k, A; k', A') = (k', A')$.
- Bellman equation:

$$V(k, A) = \max_{k' \in \Gamma(k,A)} \left\{ F(k, A; k') + \beta \int V(k', A') f(A'|A)\,\mathrm{d}A' \right\}.$$

1.2.2 Example 2: Stochastic growth with i.i.d. shocks and the cash-on-hand trick

Consider the environment from the previous example. It turns out that for a specific case it is possible to reduce the dimensionality of the state space to one. This case is the one where the A_t is i.i.d. The question that we want to answer is whether A should be a part of the state or not. Insight: only 'cash-on-hand' is relevant as a state. Once we know A_t, given that we know k_t, we can compute cash-on-hand as

$$x_t = A_t k_t^\alpha + (1-\delta)k_t.$$

In this case, the dynamic programming form of this problem is given by

- State: x,
- Control: k',
- Feasible set correspondence: $\tilde{\Gamma}(x) = [0, x]$,

- Return function: $F(x; k') = \ln(x - k')$,
- Law of motion: $x' = \tilde{h}(x; k', A') = A'k'^{\alpha} + (1-\delta)k'$,
- Bellman equation:

$$\tilde{V}(x) = \max_{k' \in \tilde{\Gamma}(x)} \left\{ \tilde{F}(x; k') + \beta \int \tilde{V}(\tilde{h}(x; k', A')) f(A') \, \mathrm{d}A' \right\}.$$

Note that as the conditional density of A' equal the unconditional density by the i.i.d. assumption. the expectation in the Bellman equation is correct. We thus see that cash-on-hand gives us the full information about the economic environment at t and that the state can be condensed in this case.

1.2.3 Example 3: Savings with stochastic earnings

Time is discrete and finite, $t = 0, \ldots, T$. Consider the earnings process for ω_t given by

$$\omega_t = \rho_1 \omega_{t-1} + \rho_2 \omega_{t-2} + \varepsilon_t,$$

where $\varepsilon_t \sim N(0, \sigma)$. At any t, the budget constraint is given by

$$c_t + \frac{a_{t+1}}{R} \leqslant a_t + \omega_t,$$

where for simplicity we impose the no-borrowing condition $a_{t+1} \geqslant 0$, $\forall t$. The economy is populated by a representative household with objective

$$\mathbb{E}_0 \left[\sum_{t=0}^{T} \beta^t \ln(c_t) \right],$$

with a_0, w_0 and w_{-1} given. The dynamic programming form of this economy is given by

- State: a, w, w_{-1}.
- Control: a'.
- Feasible set correspondence:

$$a' \in \Gamma_t(a, \omega, \omega_{-1}) = [0, R(a + \omega)].$$

- Return function:

$$F_t(a, \omega, \omega_{-1}, a') = \ln\left(a + \omega - \frac{a'}{R}\right).$$

- Law of motion:

$$\begin{bmatrix} a' \\ \omega' \\ \omega'_{-1} \end{bmatrix} = g_t(a, \omega, \omega_{-1}; a'; \omega') = \begin{bmatrix} a' \\ \omega' \\ \omega \end{bmatrix}.$$

- Bellman equations (for $t = 0, \dots, T$)

$$V_t(a, \omega, \omega_{-1}) = \max_{a' \in \Gamma_t(a,\omega,\omega_{-1})} \left\{ \ln \left(a + \omega - \frac{a'}{R} \right) + \beta \mathbb{E} \left[V_{t+1}(a', \omega', \omega) | \omega, \omega_{-1} \right] \right\},$$

where

$$\mathbb{E} \left[V_{t+1}(a', \omega', \omega | \omega, \omega_{-1}) \right] = \int_{-\infty}^{\infty} V_{t+1}(a', \rho_1 \omega + \rho_2 \omega_{-1} + \varepsilon', \omega) f(\varepsilon') d\varepsilon',$$

and $V_{T+1}(a, \omega, \omega_{-1}) = 0$.

To obtain the Euler equation, we proceed as usual by taking the FOC of the Bellman Equation w.r.t. a'. Assuming an interior solution

$$-\frac{1}{R} \frac{1}{a + \omega - \dfrac{a'}{R}} + \beta \mathbb{E} \left[V_{1,t+1}(a', \omega', \omega) | \omega, \omega_{-1} \right] = 0,$$

where $V_{1,t+1}$ denotes the derivative of V_{t+1} w.r.t. its first argument, and thus by the envelope theorem

$$V_{1,t}(a, \omega, \omega_{-1}) = \frac{1}{a + \omega - \dfrac{a'}{R}}.$$

Therefore the Euler equation reads out as

$$\frac{1}{a + \omega - \dfrac{a'}{R}} = R \beta \mathbb{E} \left[\left. \frac{1}{a' + \omega' - \dfrac{a''}{R}} \right| \omega, \omega_{-1} \right].$$

1.2.4 Example 4: Discrete-choice McCall Search Model (a ‘real-options problem’)

We face the problem of a worker decision: which jobs to accept and when to start working. The simplest model of search frictions is given by the following environment:

- Time is discrete and infinite.
- The worker is infinitely-lived, risk-neutral, and discounts the future at a rate $\beta \in (0, 1)$.
- Each period the worker draws a wage offer from a cumulative distribution function $F(w)$.
- Draws are independent and identically distributed, with support $[0, \bar{w}]$.
- Search is undirected in the sense that the worker has no ability to direct her search towards different parts of the wage distribution (or towards different types of jobs).

- If the worker accepts the offer, she gets w forever (cannot be fired).
- If she rejects the offer, she gets unemployment benefit b in the current period, and draws w' in the next period, where $0 < b > \bar{w}$.

Dynamic programming form: We focus on the state in which the worker is still searching[4]

- Shock: w.
- State: w.
- Control: $u \in \{0, 1\}$, where 0 means reject the wage offer and 1 means accept the wage offer.
- Feasible set correspondence: $\Gamma(w) = \{0, 1\}$.
- Return function: $F(w, u) = u \cdot w + (1 - u) \cdot b$.
- Law of Motion: $w' = H(w, u; w') = w'$.

Bellman equation: Let $V^{acc}(w)$ be the value of accepting an offer w, and let $V^{rej}(w)$ be the value of rejecting an offer w. Then

$$V^{acc}(w) = w + \beta w + \beta^2 w + \cdots = \sum_{t=0}^{\infty} \beta^t w = \frac{w}{1-\beta}, \tag{2}$$

$$V^{rej}(w) = b + \beta \mathbb{E}\left[V(w')|w\right] = b + \beta \mathbb{E}\left[V(w')\right] = b + \beta \int_0^{\bar{w}} V(w') f(w') \, \mathrm{d}w'. \tag{3}$$

Therefore, let $V(w)$ be the value of having drawn an offer w (before accepting or rejecting), which is given by[5]

$$V(w) = \max_{u \in \{0,1\}} \left\{ u \frac{w}{1-\beta} + (1-u) \left[b + \beta \int_0^{\bar{w}} V(w') f(w') dw' \right] \right\}. \tag{4}$$

[4] We could have also defined employed-unemployed as a separate state. Employed state is trivial, you can try to define the problem in this way as an exercise.

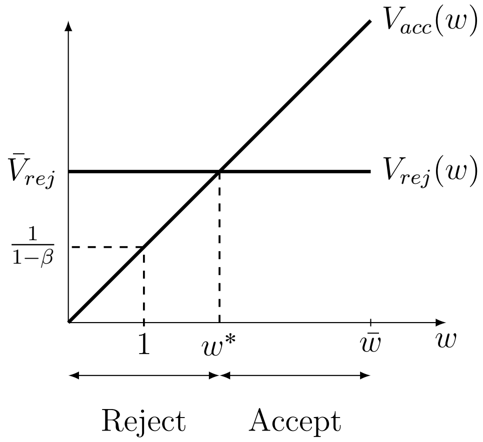
[5] Can also write it as

$$V(w) = \max\left\{V^{acc}(w), V^{rej}(w)\right\} = \max\left\{\frac{w}{1-\beta}, b + \beta \int_0^{\bar{w}} V(w') f(w') dw'\right\}.$$

Characterization of optimal policy: The optimal policy must be a cut-off rule, i.e.,

$$g(w) = \begin{cases} 1 & \text{if} \quad w \geqslant w^*, \\ 0 & \text{if} \quad w < w^*, \end{cases} \tag{5}$$

for some number $w^* \in \mathbb{R}^+$, call it the reservation wage of the worker. To compute this value, note that when drawing and offer $w = w^*$, she must be indifferent between accepting or rejecting it. Therefore, by indifference at w^* we must have



Formally,

$$\frac{w^*}{1-\beta} = V^{acc}(w^*) = \bar{V}_{rej} = V^{rej}(w^*) = b + \beta \int_0^{\bar{w}} V(w') f(w') \, \mathrm{d}w. \tag{6}$$

As the worker is infinitely-lived, she will face the same problem in the next period if she rejects. In other words, she will have to draw a new offer w', which according to the policy rule (5) she will reject if $w' < w^*$ obtaining $\bar{V}_{rej}$, and accept if $w' \geqslant w^*$. Consequently, the previous expression can be rewritten as

$$\frac{w^*}{1-\beta} = b + \beta \left[\underbrace{\int_0^{w^*} \frac{w^*}{1-\beta} f(w') \, \mathrm{d}w'}_{\text{Reject tomorrow}} + \underbrace{\int_{w^*}^{\bar{w}} \frac{w'}{1-\beta} f(w') \, \mathrm{d}w'}_{\text{Accept tomorrow}} \right]. \tag{7}$$

To obtain w^*, add and subtract

$$\int_{w^*}^{\bar{w}} \frac{w^*}{1-\beta} f(w') \, \mathrm{d}w',$$

to the right-hand side of the previous equation obtaining

$$\frac{w^*}{1-\beta} = b + \beta \left[\int_0^{\bar{w}} \frac{w^*}{1-\beta} f(w') dw' + \int_{w^*}^{\bar{w}} \frac{w' - w^*}{1-\beta} f(w') dw' \right]$$

Simplyifying yields

$$\underbrace{w^* - b}_{\equiv g(w^*)} = \underbrace{\beta \int_{w^*}^{\bar{w}} \frac{w' - w^*}{1-\beta} f(w')\,\mathrm{d}w'}_{\equiv h(w^*)}, \tag{W}$$

where $g(w^*)$ denotes the cost of searching one more time after drawing an offer w^* and $h(w^*)$ the expected discounted benefit of searching one more time (and obtaining a higher offer $w' > w^*$). Note that $g(0) = -b$, $g'(w) > 0$, $h(\bar{w}) = 0$, and by Leibniz rule

$$h'(w^*) = -\frac{w^* - w^*}{1-\beta} f(w^*) + \int_{w^*}^{\bar{w}} -\frac{1}{1-\beta} f(w')\,\mathrm{d}w' < 0.$$

Therefore, (W) has a unique solution w^*.

Comparative statics:

- b $\uparrow \implies g(w^*)$ shifts down, $h(\cdot)$ unaffected $\implies w^* \uparrow$,
- $\beta \uparrow h(\cdot)$ shifts up, $g(w^*)$ unaffected $\implies w^* \uparrow$.

One can evaluate $\frac{\mathrm{d}w^*}{\mathrm{d}b}$ and $\frac{\mathrm{d}w^*}{\mathrm{d}\beta}$ using the Implicit Function Theorem.[6] Consider the following equilibrium equation

$$E(x^*(\alpha), \alpha) = 0, \tag{E}$$

where $\alpha \in \mathbb{R}$ are parameters, and $x^* : \mathbb{R} \to \mathbb{R}$ is a policy or outcome function of interest. Totally differentiating (E) yields

$$E_x(x^*(\alpha), \alpha)\,\mathrm{d}x^* + E_\alpha(x^*(\alpha), \alpha)\,\mathrm{d}\alpha = 0,$$

so that for infinitesimal $\mathrm{d}x^*$, $\mathrm{d}\alpha$

$$x^{*\prime}(\alpha) = \frac{\mathrm{d}x^*}{\mathrm{d}\alpha} = -\frac{E_\alpha(x^*(\alpha), \alpha)}{E_x(x^*(\alpha), \alpha)}.$$

[6] Homework: Find $\frac{\mathrm{d}w^*}{\mathrm{d}b}$ and $\frac{\mathrm{d}w^*}{\mathrm{d}\beta}$. Can you say something about > 0, < 0, > 1 or < 1?

2 Euler Equations in Stochastic Framework

2.1 General framework

Time is discrete and infinite. Let z_t for $t = 1, 2, \ldots$ denote a sequence of shocks to an economy, drawn from a finite set. This process may have an arbitrary probability distribution over time. We will denote histories of shocks up to t as

$$z^t \equiv (z_1, z_2, \ldots, z_t),$$

and the set of all possible histories of length t by Z^t. As an example, take a process z_t that can only take two values from the set $S = \{\underline{z}, \bar{z}\}$ and consider the specific history

$$z^3 = (\bar{z}, \underline{z}, \bar{z}).$$

Note that $z^3 \in Z^3 = Z \times Z \times Z$. Usually we will want to to refer to a sub-history of a history z^t. For example, to refer to the history of shocks in z^t up to time $t-1$, we write $z^t_{\to t-1}$. In our example, we would have

$$z^3_{\to 2} = (\bar{z}, \underline{z}).$$

To refer to a single shocks z_k in a given history z^t we use sub-indexes. For example,to read off the last shock in a history z^t we write z^t_t. In the example, this would be ç

$$z^3_3 = \bar{z}.$$

The probability that a history z^t occurs is denoted by $\pi_t(z^t)$. We may view $\{\pi_t(\cdot)\}_{t=1}^{\infty}$ as a sequence of functions mapping from the set $\{s^t\}$ of possible histories at t to $\mathbb{R}$. This sequence of probability functions fulfills the following consistency requirements:

$$\sum_{z^t} \pi_t(z^t) = 1, \quad \text{for all } t,$$
$$\sum_{z^{t+1}: z^{t+1}_{\to t} = z^t} \pi_{t+1}(z^{t+1}) = \pi_t(z^t), \quad \text{for all } s^t, \text{ for all } t.$$

The first requirement says that the unconditional probabilities of all histories at must sum up to one at any t. The second says that the sum of probabilities following a particular node in the event tree must equal the probability of reaching that node. Conditional probabilities are given by

$$\Pr(z_{t+1} = z' | z^t) = \frac{\pi_{t+1}((z^t, z'))}{\pi_t(z^t)}.$$

The policy function at time t (i.e. choices made by the agents at t) is conditioned on information at time t. Formally, they are functions defined on the set oh histories at that point: $g_t : \{s^t\} \to \mathbb{R}$. This requirement ensures that 'agents cannot see into the future'.[7] The objective of the representative agent of the economy is to choose a sequence of policy functions $\{g_t(\cdot)\}$ in order to maximize

$$\max_{\{g_t(\cdot)\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \sum_{z^t \in Z^t} \pi_t(z^t) F_t\left(z^t, \{g_\tau(z^t_{\to\tau})\}_{\tau=0}^{t}\right), \tag{8}$$

where $F_t(\cdot)$ is the return function at t, which we allow to depend on the history of the shock and all decisions taken along this history up to t.

2.2 Examples

2.2.1 The stochastic consumption-savings model

Consider a standard consumption-savings problem where the wage at any t, z_t, can take only two values $z_t \in Z = \{\underline{z}, \bar{z}\}$. The transition probability between wages from t to $t+1$ is given by $\Pr(z_{t+1} = \tilde{z} | z_t = \tilde{z}) = 2/3$. As before, we will denote the set of all possible histories of length t as Z^t. We will look for policy functions $c_t(z^t)$ and $a_{t+1}(z^t)$ such that

$$\begin{aligned} c_t :&\quad Z^t \to \mathbb{R}_0^+, \\ a_{t+1} :&\quad Z^t \to \mathbb{R}_0^+. \end{aligned}$$

Note that consumption for period t and assets for period $t+1$ are chosen in period t, so both are policy functions at t and we write $c_t(z^t)$ and $a_{t+1}(z^t)$. The budget constraint of the representative agent at node z^t is

$$a_{t+1}(z^t) = R[z^t_t + a_t(z^t_{\to t-1}) - c_t(z^t)],$$

where we impose an exogenous constant no-borrowing limit $\underline{a}$, i.e. $a_{t+1}(z^t) \geqslant \underline{a}$, $\forall z^t, t$. Recall that z^t_t denotes the last element of history z^t and $a_t(z^t_{\to t-1})$ refers to all the history up to t, i.e., the previous sub-history ending up in node z^t.

We can state the maximization problem as

$$\begin{aligned} \max_{\left\{(c_t(z^t),\ a_{t+1}(z^t))_{z^t \in Z^t}\right\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t \underbrace{\sum_{s^t \in S^t} \pi_t(z^t) u\left(c_t(z^t)\right)}_{\mathbb{E}_0[u(c_t)]} \\ s.t. \quad & a_{t+1}(z^t) \leqslant R\left[z^t_t + a_t(z^t_{\to t-1}) - c_t(z^t)\right], \quad \forall z^t \in Z^t, \quad \forall t, \\ & a_{t+1}(z^t) \geqslant \underline{a}, \quad \forall z^t \in Z^t, \quad \forall t, \\ & a_0 \text{ given.} \end{aligned}$$

[7] In measure-theoretic terms, one would say that these functions are measurable with respect to the filtration $\mathcal{F}_t$ created by the shock history s^t

Note that as $u'(\cdot) > 0$, the budget constraint will hold with equality for any $z^t \in Z^t$ and $\forall t$. To solve this model we set up the following Lagrangean

$$
\begin{aligned}
&\mathscr{L}\left(\left\{\left(c_t(z^t), a_{t+1}(z^t), \lambda_t(z^t), \mu_t(z^t)\right)_{z^t \in Z^t}\right\}_{t=0}^{\infty}\right) = \\
&\qquad = \sum_{t=0}^{\infty} \left\{ \beta^t \sum_{z^t \in Z^t} \pi_t(z^t) u\left(c_t(z^t)\right) + \cdots \right. \\
&\qquad\qquad \cdots + \sum_{z^t \in Z^t} \lambda_t(z^t) \left[R\left(z_t^t + a_t(z^t_{\to t-1}) - c_t(z^t)\right) - a_{t+1}(z^t)\right] + \cdots \\
&\qquad\qquad \left. \cdots + \sum_{z^t \in Z^t} \mu_t(z^t) \left[a_{t+1}(z^t) - \underline{a}\right] \right\}
\end{aligned}
$$

The F.O.C. necessary conditions yield [8]

$$
\begin{aligned}
\frac{\partial \mathscr{L}(\cdot)}{\partial c_t(z^t)} &= \beta^t \pi_t(z^t) u'\left(c_t(z^t)\right) - \lambda_t(z^t) R = 0, \quad \forall z^t \in Z^t, \quad \forall t, \\
\frac{\partial \mathscr{L}(\cdot)}{\partial a_{t+1}(z^t)} &= -\lambda_t(z^t) + \mu_t(z^t) + \sum_{z^{t+1} : z^{t+1}_{\to t} = z^t} R \lambda_{t+1}(z^{t+1}) = 0, \quad \forall z^t \in Z^t, \quad \forall t.
\end{aligned}
$$

Combining both conditions we obtain

$$
\beta^t \pi_t(z^t) u'\left(c_t(z^t)\right) - \mu_t(z^t) = R \sum_{z^{t+1} : z^{t+1}_{\to t} = z^t} \beta^{t+1} \pi_{t+1}(z^{t+1}) u'\left(c_{t+1}(z^{t+1})\right).
$$

Dividing both sides by $\beta^t \pi_t(z^t)$ yields

$$
u'\left(c_t(z^t)\right) \geqslant \underbrace{R\beta \sum_{z^{t+1} : z^{t+1}_{\to t} = z^t} \underbrace{\frac{\pi_{t+1}(z^{t+1})}{\pi_t(z^t)}}_{\Pr(z_{t+1} | z_t)} u'\left(c_{t+1}(z^{t+1})\right)}_{\text{Conditional expectation of mg. utility}}, \quad \forall z^t \in Z^t, \quad \forall t,
$$

which is the Euler equation with the usual interpretation.[9] Note that the Euler equation holds with equality as long as $a_{t+1}(z^t) > \underline{a}$ (unconstrained). Otherwise it holds with $>$ and the consumer is constrained, choosing $a_{t+1}(z^t) = \underline{a}$. Restricting to the interior case, we write

$$
\begin{aligned}
u'\left(c_t(z^t)\right) &= R\beta \mathbb{E}\left[u'\left(c_{t+1}(z^{t+1})\right) | z^t\right] \\
&= R\beta \mathbb{E}_t\left[u'\left(c_{t+1}(z^{t+1})\right)\right], \quad \forall z^t \in Z^t, \quad \forall t,
\end{aligned}
$$

[8] Note that in the second F.O.C., the summation includes all the possible histories z^{t+1} that until time t have the same past as z^t, that is, we sum over the continuation of a particular node in the event history tree.

[9] Of course, we could substitute the values for the conditional probabilities π_{t+1}/π_t, but we will stay with the more general notation because it carries over to other settings.

where the second line is just different notation but means exactly the same as the first line.[10] In most papers and books we can find it written in short-hand notation as

$$u'(c_t) = R\beta \mathbb{E}_t [u'(c_{t+1})], \quad \forall t,$$

where the dependence of c_t, c_{t+1} and k_{t+1} on histories is understood.

2.2.2 The stochastic neo-classical growth model

Consider a neo-classical growth economy with production function

$$y_t = z_t F(k_t),$$

where z_t is an i.i.d. productivity shock that takes a low value $\underset{\sim}{z} > 0$ with probability 0.5 and a high values $\bar{z} > \underset{\sim}{z}$ with probability 0.5 each period. Therefore, the probability functions $\pi_t(\cdot)$ have the following properties:

$$\begin{aligned} \pi_t(z^t) &= 0.5^t, \quad \text{for all } z^t, t, \\ \Pr(z_{t+1} = \underset{\sim}{z}|z^t) = \Pr(z_{t+1} = \bar{z}|z^t) &= 0.5, \quad \text{for all } z^t, t. \end{aligned}$$

The capital stock for period $t+1$ is chosen in period t, so it is a policy function at t and we write $k_{t+1}(z^t)$. Also consumption is decided at t, so we write $c_t(z^t)$. The feasibility constraint for the agent at node z^t is

$$k_{t+1}(z^t) \leqslant \underbrace{z_t^t F(k_t(z^t_{\to t-1})) + (1-\delta)k_t(z^t_{t-1})}_{\equiv f(k_t(z^t_{\to t-1}), z_t)} - c_t(z^t).$$

Under our usual regularity assumptions about $u(\cdot)$, this constraint will always hold with equality, thus we can write the criterion on this problem as

$$U = \sum_{t=0}^{\infty} \beta^t \sum_{z^t \in Z^t} \pi_t(z^t) u\left(f(k_t(z^t_{\to t-1}), z_t) - k_{t+1}(z^t)\right).$$

In an event-tree figure we can easily see that the choice $k_{t+1}(z^t)$ affects utility at the node z^t and the two subsequent nodes z^{t+1} that follow up. So the first-order condition with respect to $k_{t+1}(z^t)$, for any $z^t \in Z^t$ and any t is

$$\begin{aligned} \frac{\partial U}{\partial k_{t+1}(z^t)} = -\beta^t \pi_t(z^t) u'(c_t(z^t)) + \cdots \\ \cdots + \sum_{z^{t+1}: z^{t+1}_{\to t} = z^t} \beta^{t+1} \pi_{t+1}(z^{t+1}) u'(c_{t+1}(z^{t+1})) f_k(k_{t+1}(z^{t+1}_{\to t}), z_{t+1}) = 0. \end{aligned}$$

[10] Note that this is just the definition of conditional expectations: $\mathbb{E}_t[h(z^{t+k})] \equiv \mathbb{E}[h(z^{t+k})|z^t]$ for any function $h(\cdot)$, $k > 0$ and some stochastic process z_t.

Dividing by $\pi_t(z^t)$ we obtain

$$u'(c_t(z^t)) = \beta \sum_{z^{t+1}:z^{t+1}_{\to t}=z^t} \frac{\pi_{t+1}(z^{t+1})}{\pi_t(z^t)} u'(c_{t+1}(z^{t+1})) f_k(k_{t+1}(z^{t+1}_{\to t}), z_{t+1}), \quad \forall z^t \in Z^t, \quad \forall t,$$

where we recognize the conditional probabilities in the fractions π_{t+1}/π_t.[11] Now we can bring the Euler equation into its typical form, which is

$$\begin{aligned} u'(c_t(z^t)) &= \beta \mathbb{E}\left[u'(c_{t+1}(z^{t+1})) f_k(k_{t+1}(z^{t+1}_{\to t}), z_{t+1}) | z^t\right] \\ &= \beta \mathbb{E}_t\left[u'(c_{t+1}(z^{t+1})) f_k(k_{t+1}(z^{t+1}_{\to t}), z_{t+1})\right], \quad \forall z^t \in Z^t, \quad \forall t. \end{aligned}$$

The intuition for this Euler equation is straightforward: the marginal utility loss from investing one more unit at node z^t must equal the marginal discounted expected gain, which is the marginal increase in productivity times marginal utility of consumption at the respective nodes at $t+1$.

[11] Of course, we could simplify the conditional probabilities due to the i.i.d. assumption, but we will stay with the more general notation because it carries over to other settings.

3 Recursive Competitive Equilibrium

3.1 Environment

- Production function: $y_t = A_t F(K_t, L_t)$, where A_t follows a first-order Markov process.
- Measure 1 of identical households, indexed by $i \in [0,1]$. They can save in capital and work $l_t^{(i)} \in [0,1]$. The budget constraint (for each household), is given by

$$c_t^{(i)} + k_{t+1}^{(i)} \leqslant w_t l_t^{(i)} + (1 - \delta + r_t) k_t^{(i)}, \quad \forall i, \quad \forall t. \tag{BC}$$

  Remark. *In this setting, we will look for a symmetric equilibrium where every single agent does the same (but we are not imposing that). Thus we want to obtain*

$$k_t^{(i)} = k_t, \quad l_t^{(i)} = l_t, \quad c_t^{(i)} = c_t, \quad \forall i, \quad \forall t,$$

  as the equilibrium outcome.

- Aggregation [12]

$$K_t = \int_0^1 k_t^{(i)} di = \int_0^1 k_t di = k_t, \quad \forall t,$$

  where the second equality comes from the symmetric equilibrium allocation. This is known as the 'big-K, little-k trick'. Similarly, we also have

$$L_t = \int_0^1 l_t^{(i)} di = \int_0^1 l_t di = l_t \quad \forall t.$$

- Central idea of recursive competitive equilibrium:
  - Prices are a function of the economy's state (not the entire history).
  - Define individual rationality from the Bellman equations instead of sequence problems.
- State:[13] $X_t = (A_t, K_t)$.
- Prices: $r_t = r(X_t)$, $w_t = w(X_t)$, i.e. $r_t = r(A_t, K_t)$, $w_t = w(A_t, K_t)$.

[12] Note that one agent cannot move K_t or L_t on their own by choosing $(k_t^{(i)}, l_t^{(i)})$ as they are atomistic agents.

[13] Note that L_t is not a state, it is decided in each period, as consumption C_t.

3.2 Firm's problem

Price-taking behaviour. Profit maximization

$$\max_{\{k^d, l^d\}} AF(k^d, l^d) - r(X)k^d - w(X)l^d.$$

In equilibrium

$$r(X) = AF_k(k^d, l^d), \tag{r}$$

$$w(X) = AF_l(k^d, l^d), \tag{w}$$

and the firm makes zero profits, thus

$$AF\left(k^{d*}, l^{d*}\right) = r(X)k^{d*} + w(X)l^{d*}.$$

3.3 Household's problem

- Agents have a perceived law of motion for K given by

$$K' = G(A, K).$$

  Again, in this environment agents can't directly modify K by choosing k.

- Given their perceived law of motion, agents can forecast prices

$$r' = r(A', K'),$$
$$w' = w(A', K').$$

- State for the individual: k (individual capital stock), A (TFP) and K (aggregate capital stock).

3.3.1 Bellman Equation

$$\begin{aligned} V\underbrace{(A, K; k)}_{\text{state}} &= \max_{c,l,k'} \left\{u(c, 1-l) + \beta\mathbb{E}\left[V\left(A', G(A, K); k'\right)|A\right]\right\} \\ \text{s.t. } & c + k' \leqslant w(A, K)\, l + (1 - \delta + r(A, K))\, k \end{aligned} \tag{BE}$$

with decision rules (or policy functions)

$$\left.\begin{aligned} c^* &= g^c(A, K; k) \\ l^* &= g^l(A, K; k) \\ k'^* &= g^{k'}(A, K; k) \end{aligned}\right\} \tag{g}$$

The state in the Bellman equation[14] is given by $(A, K; k)$, where in principle we allow households to deviate from others by choosing their own k.

[14] The Bellman equation is given by both the equation and the constraint.

3.3.2 Rational Expectations

In this model Rational Expectations imply that the perceived Law of Motion is equal to the realized law of motion, i.e.

$$K' = G(A, K) = \int_0^1 k_t'^{(i)} di = \underbrace{k'^* = g^{k'}(A, K; K)}_{\text{Policy function}},$$

where we use the 'big-K, little-k' trick, and we substitute $k = K$ in the policy function because we are in a symmetric equilibrium. Thus $K' = k'^*$ is implied by rational expectations.

3.4 Equilibrium

Definition 3.1 (RCE). *A Recursive Competitive Equilibrium (RCE) consists of a value function $V(A, K; k)$ and policy functions $\{g^c(A, K; k), g^l(A, K; k), g^{k'}(A, K; k)\}$ for the household, policy functions $\{k^d(A, K), l^d(A, K)\}$ for the firm, a law of motion $G(A, K)$ and pricing functions $r(A, K), w(A, K)$ such that:*

- *$\{k^d(A, K), l^d(A, K)\}$ maximize firm's profits given $r(A, K), w(A, K)$ for all (A, K),*
- *$\{V(A, K; k), g^c(A, K; k), g^l(A, K; k), g^{k'}(A, K; k)\}$ solve the household's problem, i.e. (BE), given $r(A, K), w(A, K), G(A, K)$ for all (A, K),*
- *expectations are rational, i.e., $G(A, K) = g^{k'}(A, K; K)$,*
- *markets clear:*

$$l^d(A, K) = g^l(A, K; K), \quad \forall (A, K),$$

$$k^d(A, K) = K, \quad \forall (A, K),$$

$$g^c(A, K; K) + g^{k'}(A, K; K) = AF(K, \underbrace{g^l(A, K; K)}_{=l^d(A,K)}) + (1 - \delta)K, \quad \forall (A, K).$$

Let's derive the Euler Equation of this problem. The F.O.C. w.r.t. k' in (BE) (in equilibrium) yields

$$- u_c \left(g^c(A, K; k), 1 - g^l(A, K; k)\right) + \beta \mathbb{E}\left[V_k\left(A', G(A, K); g^{k'}(A, K; k)\right) | A\right] = 0. \quad (9)$$

Note that k' does not impact K' as the agent is atomistic.[15] As usual, by the envelope theorem we have

$$V_k(A, G(A, K); k) = \frac{\partial u(c, 1 - l)}{\partial k},$$

[15]Important not to make the household a monopolist who can manipulate prices!

Knowing that the budget constraint (BC) will hold with equality we can substitute out c in the previous equation obtaining

$$\begin{aligned} V_k\left(A, G\left(A, K\right); k\right) &= \frac{\partial u\left(w\left(A, K\right) l + \left(1-\delta+r(A, K)\right) k - k', 1-l\right)}{\partial k} \\ &= \left(1-\delta+r\left(A, K\right)\right) u_c(c, 1-l). \end{aligned} \tag{10}$$

Combining (9) and (10) we can rewrite the F.O.C. of (BE) as

$$u_c\left(g^c\left(A, K; k\right), 1-g^l\left(A, K; k\right)\right) = \beta\mathbb{E}\left[\left(1-\delta+r\left(A', K'\right)\right) u_c(g^c\left(A, K; k\right), 1-g^l\left(A, K; k\right))|A\right],$$

where in the equilibrium we also know that

$$r\left(A', K'\right) = A' F_K\left(G\left(A, K\right), L'\right). \tag{11}$$

Then

$$\begin{aligned} &u_c\left(\underbrace{g^c\left(A, K; K\right)}_{c_t^*}, 1-\underbrace{g^l\left(A, K; k\right)}_{l_t^*}\right) = \\ &\quad = \beta\mathbb{E}\left[\underbrace{\left(1-\delta+A' F_K\left(G\left(A, K\right), L'\right)\right)}_{MPK_{t+1}} u_c(\underbrace{g^c\left(A', G(A, K); K'\right)}_{c_{t+1}^*}, 1-\underbrace{g^l\left(A', G(A, K); K'\right)}_{l_{t+1}^*}) \middle| A\right] \end{aligned}$$

You may check that this is the same equation that you can obtain when solving the planner's problem, thus the allocations prescribed by the RCE are exactly the same as the ones given by a planner. As a consequence, RCE is efficient.